Alternative Proof of the Erdős Sumset Conjecture

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ALTERNATIVE PROOF OF THE ERDŐS SUMSET CONJECTURE

Abstract. In this thesis, we will discuss an ergodic proof of the Sumset Conjecture of Erdős, which asks if every set $A \subseteq \mathbb{N}$ with positive upper Banach density contains $B + C$ for some $B, C \subseteq \mathbb{N}$ infinite. This result was originally proved by Moreira, Richter, and Robertson in 2019 using ultrafilters, however in this proof we will adapt the method of progressive measures recently developed by Kra, Moreira, Richter, and Robertson. We closely follow their proof, simplifying what we can along the way.

1. Introduction

Szemerèdi's Theorem states that any sufficiently dense set of the natural numbers must contain arbitrarily long arithmetic progressions [Sze75]. This was proved in 1975 using combinatorial methods, and was originally conjectured by Erdős and Turàn in 1936. However, in 1977, Furstenberg provided a new proof of Szemerédi's Theorem using ergodic theory and the structure theory of measure-preserving systems in [Fur77], wherein he developed the Furstenberg Correspondence Principle, which allowed for certain combinatorial statements to be rephrased as dynamical statements. This intersection between dynamics and combinatorics has been quite fruitful, improving our understanding of the general structure of dynamical systems, as well as solving various combinatorial questions.

Szemerèdi's Theorem was conjectured after Van der Waerden proved in 1927 that for any finite coloring of the natural numbers, then there is a color containing arbitrarily long arithmetic progressions [vdWBL27]. Szemerèdi's Theorem can be thought of as the "density analogue" to Van der Waerden's theorem, asking if instead the assumption that a subset $A \subset \mathbb{N}$ having positive density in $\mathbb{N}$, a notion defined in the next section, is enough for A to contain arbitrarily long arithmetic progressions.

Hindman's Theorem states that for any finite coloring of the natural numbers, one of the colors must contain an IP set [Hin74]. Given an infinite set $B \subset \mathbb{N}$, the IP set corresponding to B is all finite sums of distinct elements of B. In much the same way that Szemerèdi is a density analogue of Van der Waerden, a density analogue of Hindman was also sought after. Due to a counterexample of Strauss, it is known that not every positive density set contains an IP set, see [Hin79]. However, something close was achieved last year by Kra, Moreira, Richter, and Robertson.

The Density Finite Sums Theorem proved in [KMRR26] gives the best analogue known thus far, that for any positive density set A and for each k, there is an infinite set B_k whose sums of distinct elements of length *at most* k are contained in (possibly a shift of) A. Notice that Hindman supplies us with a color and an infinite set B so that the IP set for B is contained in that color, while this density analogue provides us only with an infinite set that depends on k, whose sums up to length k land in A, or a shift of it. In the density analogue, we do not have an IP set contained in A, rather for each k we can get a "truncated" IP set.

The question we will answer in this thesis is the Erdős Sumset Conjecture: Given a subset $A \subset \mathbb{N}$ of positive density, does there exist a pair of infinite sets $B, C \subset \mathbb{N}$ so that $B + C := \{b + c : b \in B, c \in C\} \subset A$? This was proved in 2019 by Moreira, Richter, and Robertson using ultrafilters [MRR19], and a generalization of it was later proved by Kra, Moreira, Richter, and Robertson using ergodic theory [KMRR23], specifically using the theory of cube measures. This conjecture immediately follows from the Density Finite Sums Theorem, however the purpose of this thesis was to learn more about this area by recreating the proof of the sumset conjecture using the techniques developed in [KMRR26] and to see what can be simplified along the way, and to illuminate the main ideas of the newer work in a simpler situation.

The organization in this work is very similar to that of [KMRR26], we will collect some useful results from ergodic theory, as well as set up and prove the correspondence principle in the preliminary section. In the third section, we will define and discuss what progressive measures are and how they relate to the problem at hand, completing the setup of the problem and essentially reducing it to the existence of progressive measures. In the fourth section, we will produce a measure that will be shown to be progressive, as well as some of its properties. In the fifth and final section, we will carry out the technical work to show that our candidate measure is indeed progressive, completing the proof.

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We now formulate and state the main result.

Definition 1.1. For a subset $A \subset \mathbb{N}$, we define the positive upper Banach density of A, denoted $d^*(A)$, as

$$d^*(A) := \lim_{N\to\infty} \sup_{M\in\mathbb{N}} \frac{|A \cap \{M+1, \dots, M+N\}|}{N}.$$

Theorem 1.2. *If $A \subset \mathbb{N}$ has $d^*(A) > 0$, then there exist infinite subsets $B, C \subset \mathbb{N}$ with*

$$B + C = \{b + c : b \in B, c \in C\} \subset A.$$

2. Ergodic Preliminaries

Ergodic theory and measure theory: We will state several definitions and clarify the notation we are using. A compact metric space X and a homeomorphism $T : X \to X$ (X, T) is a *topological dynamical system.* A Borel probability measure μ on X is said to be *T-invariant* if $\mu(T^{-1}A) = \mu(A)$ for every Borel set $A \subset X$, and the tuple (X, μ, T) is a *measure-preserving system*, abbreviated sometimes as m.p.s (measure-preserving system). An m.p.s is said to be *ergodic* if $T^{-1}A = A$ implies $\mu(A) = 0, 1$. For $f \in L^2(X, \mu)$, we denote by Tf the function $Tf(x) = f(Tx)$. If $Tf = f$ μ-almost everywhere, we say f is *T-invariant*, and if $Tf = \lambda f$ μ-almost everywhere for some $\lambda \in \mathbb{C}$, then we say f is an eigenfunction of T with eigenvalue λ.

For a sequence of finite subsets of $\mathbb{N}$, $\Phi = (\Phi_N)_{N\in\mathbb{N}}$, we say Φ is a Følner sequence if

$$\lim_{N\to\infty} \frac{|(\Phi_N + 1) \cap \Phi_N|}{|\Phi_N|} = 1.$$

For a measure preserving system (X, μ, T), $f \in L^2(X, \mu)$, the average below converges in $L^2(X, \mu)$ to

$$\frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f \circ T^n \to \mathbb{P}_T f,$$

where $\mathbb{P}_T$ is the projection to T-invariant functions. When the system is ergodic, $\mathbb{P}_T f = \int f d\mu$. This is the *mean ergodic theorem*, see Chapter 2 of [EW10] for the statement for the average $\frac{1}{N}\sum_{n=0}^{N-1}$ and Chapter 8 for the general Følner statement.

Let $\mathcal{M}(X)$ denote the set of probability measures on X, note that $\mathcal{M}(X)$ is compact when equipped with the weak-* topology. For a continuous map $S : X \to X$, $\mu \in \mathcal{M}(X)$, denote by $S_*\mu$ the pushforward measure of μ by S, $S_*\mu(A) = \mu(S^{-1}A)$. Given a topological system (X, T), the set of T-invariant probability measures on X, $\mathcal{M}^T(X)$, is also compact and nonempty, see the fourth chapter of [EW10]. We will Theorem 4.1 from [EW10] frequently, namely that if $\{\nu_n\} \subset \mathcal{M}(X)$, and if Φ is a Følner sequence, then any subsequential limit of the sequence

$$(\star) \qquad \mu_N := \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T)^n_* \nu_n$$

will be in $\mathcal{M}^T(X)$, note such a limit exists by compactness.

Sketch of Proof of Theorem 4.1 of [EW10]. Let μ be a subsequential limit of the sequence $(\star)$, and let $f \in C(X)$. We will show $\int f d\mu = \int f \circ T d\mu$, this is equivalent to μ being T-invariant by a standard density argument. Then pick $N \gg 1$, it follows that we can approximate

$$\left| \int f d\mu - \int f \circ T d\mu \right|$$

by

$$\left| \int f d\mu_N - \int f \circ T d\mu_N \right|$$

by simply applying the triangle inequality twice. But this term vanishes, using the definition of μ_N and pushforward, we get

$$\begin{aligned}\left|\int f d\mu_N - \int f\circ T d\mu_N\right| &= \left|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int f\circ T^n d\nu_n - \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int f\circ T^{n+1} d\nu_n\right| \\ &= \left|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\int f\circ T^n d\nu_n - \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N+1}\int f\circ T^{n} d\nu_n\right| \\ &= \left|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N\triangle(\Phi_N+1)}\int f\circ T^n d\nu_n\right| \leq \|f\|_\infty\frac{|\Phi_N\triangle(\Phi_N+1)|}{|\Phi_N|}.\end{aligned}$$

The definition of Følner sets guarantees that this last term will vanish. ■

Remark. We include this sketch because this theorem is used so often, and because it highlights how Følner sets are great tools for constructing invariant objects. Nothing is special about this proof over $\mathbb{N}$, Følner sets can be generalized to amenable groups and used to create invariant measures in much the same way as was done above.

If (X,μ,T) and (Y,ν,S) are two measure-preserving systems and if $\pi : X\to Y$ is a surjective Borel-measurable map with $\pi\circ T = S\circ\pi$, we say Y is a *measurable factor* of X and π is a *factor map.* There is a natural correspondence between factors of measure-preserving systems and T-invariant sub σ-algebras of the Borel σ-algebra of X, see ([EW10], Lemma 6.4, Theorem 6.5). To each factor (Y,ν,S) of (X,μ,T), there is a sub σ-algebra of the Borel σ-algebra of X, call it $\mathscr{Y}$. Specifically, $\mathscr{Y} = \sigma(\pi^{-1}\mathscr{B}_Y)$, where $\mathscr{B}_Y$ is the Borel σ-algebra of Y, and for $\mathscr{F}\subset\mathcal{P}(X)$, $\sigma(\mathscr{F})$ is the smallest σ-algebra containing $\mathscr{F}$.

If $(X,\mu,T)\xrightarrow{\pi}(Y,\nu,S)$ is a factor of systems, then for $f\in L^1(X,\mu)$, we can consider the *expectation* of f against $\mathscr{Y}$, $E_\mu(f|\mathscr{Y})\in L^1(Y,\nu)$. This is the unique function in $L^1(Y,\nu)$ with the property that for every $A\in\mathscr{B}_y$,

$$\int_A E_\mu(f|\mathscr{Y})d\nu = \int_{\pi^{-1}A} f d\mu.$$

Observe that as a consequence, $E_\mu(f\circ T|\mathscr{Y}) = E_\mu(f|\mathscr{Y})\circ S$ almost everywhere. Usually we will omit the μ in E_μ, only including it to avoid confusion.

Kronecker factors of ergodic systems: ([EW10], Theorems 4.14, 6.11) Every ergodic system (X,μ,T) has a largest factor (Z_1, m_{Z_1}, R) which is isomorphic to rotation on a compact abelian group, where m_{Z_1} is the Haar measure on Z_1 and $R(z) = z+\alpha$ for some $\alpha\in Z_1$. This factor is called the Kronecker factor. Let $\mathscr{Z}_1$ denote the σ-algebra corresponding to the Kronecker factor. $\mathscr{Z}_1$ is the smallest σ-algebra with respect to which all of the T-eigenfunctions are measurable. For a compact abelian group G with Haar measure m_G, $g\in G$, the following are equivalent:

(a) Let $R_g : G\to G$ denote the map $R_g(h) = h+g$. R_g ergodic with respect to m_G.

(b) R_g uniquely ergodic. (I.e. there is only one invariant measure for R_g)

(c) $\{g^n : n\in\mathbb{Z}\}$ is dense in G.

For the Kronecker factor (Z_1, m_{Z_1}, R) of a system (X,μ,T), we will typically denote the rotation R on Z_1 by T, a slight abuse of notation.

For (X,μ,T) m.p.s, $a\in X$, we say $a\in\mathbf{gen}(\mu)$ if

$$\frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n a}\xrightarrow{*}\mu$$

in the weak-* topology. Similarly, $a\in\mathbf{gen}(\mu,\Phi)$ if

$$\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\delta_{T^n a}\xrightarrow{*}\mu.$$

Note that given any point $x\in X, \Phi$ Følner, we can find a T-invariant measure μ so that $x\in\mathbf{gen}(\mu,\Phi)$ by applying Theorem 4.1 of [EW10] to $\nu_n\equiv\delta_x$.

When (X,μ,T) is ergodic, then μ-a.e. point $x \in X$ is generic for μ, this is a consequence of the *Birkhoff pointwise ergodic theorem.* See Chapter 2 of [EW10].

Before Szemerèdi's Theorem was proved in 1975, which showed that positive density sets contain arbitrarily long arithmetic progressions, Roth proved in 1953 that arithmetic progressions of length three can always be found in positive density sets in [Rot53]. We state the dynamical version of Roth's theorem, which we will use later.

Theorem 2.1 ([EW10], Theorem 7.14)**.** *Let (X,μ,T) be a measure preserving system, $f_1, f_2 \in L^\infty(X,\mu)$, then the average*

$$\frac{1}{N}\sum_{n=1}^{N} f_1 \circ T^n \cdot f_2 \circ T^{2n}$$

converges in $L^2(X,\mu)$. In particular, if $\mu(A) > 0$, then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

Remark. While Roth's Theorem is stated above as

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0,$$

more than this is true, specifically that if Φ is a Følner sequence, then the average is still positive, i.e.

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} \mu(A \cap T^{-n}A \cap T^{-2n}A) > 0.$$

This can be seen by examining the proof of Theorem 2.1 in [EW10] and observing that the key theorems used are still valid over Følner sets. Thus we will use the result over general Følner sets freely.

Theorem 2.2 ([Fur81], Furstenberg Correspondence Principle)**.** *For any $A \subset \mathbb{N}$ of positive upper Banach density, there exists a measure-preserving system (X,μ,T), $a \in X$ with $\mu(\overline{\{T^n a : n \in \mathbb{N}\}}) = 1$ and $E \subset X$ open with $\mu(E) = d^*(A) > 0$ so that*

$$A = \{n : T^n a \in E\}.$$

Proof. Let $\chi_A \in \{0,1\}^{\mathbb{Z}}$ be the indicator function of A, namely

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}.$$

Consider the left shift map $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$, $\sigma\omega(n) = \omega(n+1)$, and form $X \coloneqq \overline{\{\sigma^n \chi_A : n \in \mathbb{Z}\}}$. Since $A \subset \mathbb{N}$ has positive upper Banach density, then there is a sequence $(M_N)_N \subset \mathbb{N}$ so that

$$\lim_{N\to\infty} \frac{|A \cap \{M_N+1,\ldots,M_N+N\}|}{N} = d^*(A).$$

Define a sequence of probability measures on X by

$$\mu_N \coloneqq \frac{1}{N}\sum_{n=M_N+1}^{M_N+N} \delta_{\sigma^n\chi_A} = \frac{1}{N}\sum_{n=M_N+1}^{M_N+N} (\sigma)^n_* \delta_{\chi_A}.$$

Take a weak-* convergent subsequence of these measures and call the limit μ. Then μ is σ-invariant, see ([EW10], Theorem 4.1). Let $E = \{x \in X : x(0) = 1\}$, E is both closed and open in X, and thus we have that $\mu_N(E) \to \mu(E)$. Observe that $\sigma^n\chi_A \in E \iff \sigma^n\chi_A(0) = 1 \iff \chi_A(n) = 1 \iff n \in A$, using this we can unpack the definition of $\mu_N(E)$ to get

$$\mu_N(E) = \frac{1}{N}\sum_{M_N+1}^{M_N+N} \delta_{\sigma^n\chi_A}(E) = \frac{1}{N}\cdot |A \cap \{M_N+1,\ldots,M_N+N\}|.$$

So $\mu_N(E) \to d^*(A)$, and as we mentioned, E being clopen implies that $\mu_N(E) \to \mu(E)$, so $\mu(E) = d^*(A) > 0$. We also have that $\sigma^n\chi_A \in E \iff n \in A$, so we have everything we sought to find. ∎

Remark. The reader is encouraged to try to use Theorem 2.2 and Theorem 2.1 to prove Roth's theorem on arithmetic progressions: If $d^*(A) > 0$, then A contains an arithmetic progression of length three, specifically that there exist $a \in A, k \in \mathbb{N}$ so that $\{a, a+k, a+2k\} \subset A$.

3. Connecting Sumsets to Dynamics: Erdős Progressions

We will now state the main dynamical definition and result we need to prove Theorem 1.2.

Definition 3.1. Let (X, μ, T) be a measure preserving system. We say $(x_0, x_1, x_2) \in X^3$ is an Erdős progression if there is an increasing sequence $c : \mathbb{N} \to \mathbb{N}$ so that $T^{c(n)}x_0 \to x_1, T^{c(n)}x_1 \to x_2$.

Our goal will be to find Erdős progressions with a prescribed starting point and an endpoint in a prescribed open set, or a preimage of it. Taking a preimage may be necessary in general, as the method of Erdős progressions was originally used in [KMRR26] to prove that for every $A \subset \mathbb{N}$ of positive upper Banach density, $k \in \mathbb{N}$, there is a $b : \mathbb{N} \to \mathbb{N}$ increasing and a $t \geq 0$ so that $\{b(i_1) + \cdots + b(i_k) : i_1 < i_2 < \cdots < i_k\} \subset A - t$. To see that the shift is necessary, consider the case $k = 2$ and $A = 2\mathbb{N} + 1$. Then for any sequence $b : \mathbb{N} \to \mathbb{N}$, there is a pair of elements $b(i), b(j)$ of the same parity, and thus their sum $b(i) + b(j)$ is even and cannot land in $2\mathbb{N} + 1$. As will be demonstrated in the proof of Theorem 1.2, we can "absorb" the shift into one of the sets B, C, so the shift present in the statement of Theorem 3.2 should be seen as a side effect of directly adapting the methods from [KMRR26].

Theorem 3.2. *For any measure preserving system (X, μ, T), $a \in X$ with $\mu(\overline{\{T^n a : n \in \mathbb{N}\}}) = 1$, $E \subset X$ open with $\mu(E) > 0$, there exists an Erdős progression of the form (a, x_1, x_2) with $x_2 \in E$ or $x_2 \in T^{-1}E$.*

It is not yet clear how Erdős progressions will lend themselves to producing sumsets. In the next lemma, we will demonstrate how Erdős progressions naturally give rise to sumsets in the return times of x_0 to neighborhoods of x_2.

Lemma 3.3. *Let (X, μ, T) be a measure preserving system, and suppose that (x_0, x_1, x_2) is an Erdős progression, and that $x_2 \in E$, where $E \subset X$ is open and $\mu(E) > 0$. Then there is an increasing sequence $b : \mathbb{N} \to \mathbb{N}$ so that*

$$\{b(i) + b(j) : i \neq j, i, j \in \mathbb{N}\} \subset \{n \in \mathbb{N} : T^n x_0 \in E\}.$$

Proof. We will produce a family of open sets in X $\{U_{i,j}\}_{i=0,1,2}^{j \in \mathbb{N}}$ and an increasing sequence $(b(n))$ with

(1) $U_{i,j}$ is an open neighborhood of x_i for every j.

(2) $T^{b(j)}x_{i-1} \in U_{i,j-1}$ for $i = 1, 2, j \in \mathbb{N}$.

Define $U_{0,j} = X, U_{2,j} = E$ for all j, and define $U_{1,0} = X,$. Since $T^{c(n)}x_1 \to x_2$, there is $n_1 \in \mathbb{N}$ so that $T^{c(n_1)}x_1 \in E$. Define $U_{1,1} \coloneqq T^{-c(n_1)}E$ and $b(1) \coloneqq c(n_1)$, $U_{1,1}$ is an open neighborhood of x_1, and $T^{b(1)}x_1 \in U_{2,0} = E, T^{b(1)}x_0 \in U_{1,0} = X$. Now there is $n_2 \in \mathbb{N}$, $n_2 > n_1$ so that both $T^{c(n_2)}x_0 \in U_{1,1}$ and $T^{c(n_2)}x_1 \in U_{2,1} = E$. Define $b(2) \coloneqq c(n_2)$ and $U_{1,2} \coloneqq U_{1,1} \cap T^{-b(2)}U_{2,1} = U_{1,1} \cap T^{-b(2)}E$, and observe that $T^{b(2)}x_0 \in U_{1,1}, T^{b(2)}x_1 \in U_{2,1} = E$.

Let $k \in \mathbb{N}$, and suppose we have chosen $\{U_{1,j}\}_{0 \leq j \leq k-1}$ and $b(1), \ldots, b(k-1)$ in accordance with (1), (2). Then $U_{1,k-1}$ is an open neighborhood of x_1, so there is $n_k \in \mathbb{N}$ with $n_k > n_{k-1} > \cdots > n_1$ and $T^{c(n_k)}x_0 \in U_{1,k-1}, T^{c(n_k)}x_1 \in E$. Define $b(k) = c(n_k)$, $U_{1,k} \coloneqq U_{1,k-1} \cap T^{-b(k)}U_{2,k-1} = U_{1,k-1} \cap T^{-b(k)}E$. Now $T^{b(k)}x_0 \in U_{1,k-1}$ and $T^{b(k)}x_1 \in U_{2,k-1} = E$, so by induction we can produce such a family of open sets and a sequence $(b(n))$ satisfying (1), (2).

Observe from the definition of the sequence $U_{1,j}$ that if $k < j$, then $U_{1,j} \subset U_{1,k}$ since $U_{1,j} = U_{1,j-1} \cap T^{-b(j)}E$. Now consider $i \neq j$, say $i < j$. Then $T^{b(j)}x_0 \in U_{1,j-1} \subseteq U_{1,i}$ since $i \leq j-1$, and since $U_{1,i} = U_{1,i-1} \cap T^{-b(i)}E$, we have that $T^{b(j)}x_0 \in T^{-b(i)}E$, i.e. $T^{b(i)+b(j)}x_0 \in E$. ■

With the connection between Erdős progressions and sumsets being more clear, we will show that Theorem 3.2 implies Theorem 1.2 with an application of the correspondence principle, and thus our new goal will be to prove Theorem 3.2.

Proof that Theorem 3.2 implies Theorem 1.2. Let $A \subset \mathbb{N}$ have positive upper Banach density, then there is a measure-preserving system (X, μ, T), a point $a \in X$ with $\mu(\overline{\{T^n a : n \in \mathbb{N}\}}) = 1$, and $E \subset X$ with $\mu(E) > 0$ with $A = \{n \in \mathbb{N} : T^n a \in E\}$. Then from Theorem 3.2, we have an Erdős progression (a, x_1, x_2) with $x_2 \in E$ or $T^{-1}E$, say $x_2 \in T^{-t}E$, $t \in \{0, 1\}$. Let $b(n)$ be the sequence guaranteed by Lemma 3.3, then we have

$$\{b(i) + b(j) : i \neq j\} \subset \{n \in \mathbb{N} : T^n a \in T^{-t}E\} = \{m \in \mathbb{N} : T^m a \in E\} - t,$$
$$\{b(i) + b(j) : i \neq j\} + t \subset \{m \in \mathbb{N} : T^m a \in E\}.$$

Define $B = \{b(2n) : n \in \mathbb{N}\}, C = \{b(2n+1)+t : n \in \mathbb{N}\}$, then for any $b \in B, c \in C$, $b = b(2n)$, $c = b(2m+1)+t$, and so $b+c = b(2n)+b(2m+1)+t \in \{b(i)+b(j) : i \neq j\}+t \subset \{m \in \mathbb{N} : T^m a \in E\}$, thus $B+C \subset \{m \in \mathbb{N} : T^m a \in E\} = A$. ■

To produce Erdős progressions, we will want to work with generic points rather than points whose orbit is dense in $\text{supp}(\mu)$ for reasons that will become apparent in the next section. We will state a new dynamical result that looks like Theorem 3.2 with generic points in place of points $a \in X$ with $\mu(\overline{\{T^n a : n \in \mathbb{N}\}}) = 1$ and show that this new theorem implies Theorem 3.2 via an ergodic decomposition argument. We will then spend the rest of the thesis proving Theorem 3.4.

Theorem 3.4. *Let (X, μ, T) be an ergodic system, Følner sequence Φ, $a \in \mathbf{gen}(\mu, \Phi)$, $E \subset X$ open with $\mu(E) > 0$. Then there is an Erdős progression of the form (a, x_1, x_2) with $x_2 \in T^{-t}E$, $t = 0, 1$.*

Proof that Theorem 3.4 implies Theorem 3.2. Let (X, μ, T) be any measure preserving system, $a \in X$ with

$$\mu(\overline{\{T^n a : n \in \mathbb{N}\}}) = 1,$$

and $E \subset X$ open with $\mu(E) > 0$. Let $Y_a = \overline{\{T^n a : n \in \mathbb{N}\}}$, then $(Y_a, \mu|_{Y_a}, T)$ is itself a measure preserving system, where $\mu|_{Y_a}(F) = \mu(F \cap Y_a)$. Define $\mu|_{Y_a} := \nu$. So let $(\nu_x)_{x \in Y_a}$ be an ergodic decomposition of ν, then there exists a ν_x ergodic with $\nu_x(E) \geq \nu(E) > 0$ since $\int_{x \in Y_a} \nu_x d\nu(x) = \nu$ by ([EW10], Theorem 6.2). Moreover, by ([Fur81], Proposition 3.9), since ν_x is ergodic for (Y_a, T), then there is a Følner sequence Φ so that $a \in \mathbf{gen}(\nu_x, \Phi)$.

Altogether, we have (Y_a, ν_x, T) ergodic, $a \in \mathbf{gen}(\nu_x, \Phi)$, and $E \cap Y_a$ open in Y_a with $\nu_x(E) > 0$, so by Theorem 3.4, there is an Erdős progression (a, x_1, x_2) with $x_2 \in T^{-t}(E \cap Y_a) \subset T^{-t}E$, so we have shown Theorem 3.2. ■

We move on to the next section where we introduce the notion of a progressive measure. Their purpose will be to produce Erdős progressions in sets of positive measure.

4. Progressive Measures

Given a topological dynamical system (X, T), we will denote the diagonal transformation $T \times T \times \cdots \times T$ on $X \times X \times \cdots \times X$ by T_Δ.

Definition 4.1. Let (X, T) be a topological dynamical system. A measure $\nu \in \mathcal{M}(X^3)$ is progressive if $\nu(\{a\} \times X^2) = 1$ for some $a \in X$, and if for any $U_1, U_2 \subset X$ open with $\nu(X \times U_1 \times U_2) > 0$, then there are infinitely many n so that

$$\nu((X \times U_1 \times U_2) \cap T_\Delta^{-n}(U_1 \times U_2 \times X)) > 0. \tag{4.1.1}$$

Lemma 4.2. *Let (X, T) be a topological dynamical system. If ν is a progressive measure, $a \in X$ with $\nu(\{a\} \times X^2) = 1$, then for any $U_1, U_2 \subset X$ open with $\nu(X \times U_1 \times U_2) > 0$, there is an Erdős progression of the form (a, x_1, x_2) with $x_i \in U_i$ for $i = 1, 2$.*

Proof. Let $U_1, U_2 \subset X$ be open sets with $\nu(X \times U_1 \times U_2) > 0$. Then there are infinitely many n satisfying (4.1.1). We will construct, using (4.1.1) at each step, a particular decreasing family of open sets in X^2 with decreasing diameter and of positive $\pi_*\nu$-measure, where $\pi : X^3 \to X^2$ is given by $\pi(x_1, x_2, x_3) = (x_2, x_3)$. After taking the intersection of this family, we will arrive at a singleton set which will form our Erdős progression in conjunction with a.

Define $V_0 = U_1 \times U_2$. To define V_1, let n_1 be such that $\nu((X \times V_0) \cap T_\Delta^{-n_1}(V_0 \times X)) > 0$. Then there is some $(a, y_1, y_2) \in \text{supp}(\nu) \cap ((X \times V_0) \cap T_\Delta^{-n_1}(V_0 \times X))$, so define V_1 to be an open ball around (y_1, y_2) with

$\operatorname{diam}(V_1) \leq \frac{1}{2}\operatorname{diam}(V_0)$ and $\{a\} \times V_1 \subset (X \times V_0) \cap T_\Delta^{-n_1}(V_0 \times X)$. It follows that $\nu(X \times V_1) > 0$ since $X \times V_1$ is an open ball around $(a, y_1, y_2) \in \operatorname{supp}(\nu)$. We will define the rest of the open sets in the same way by induction.

Suppose we have already chosen $V_0, \ldots, V_{k-1}$ and $n_1 < \cdots < n_{k-1}$ so that for $j = 0, \ldots, k-2$,

(a) $\{a\} \times V_{j+1} \subset (X \times V_j) \cap T_\Delta^{-n_{j+1}}(V_j \times X)$.

(b) $\nu(X \times V_{j+1}) > 0$.

(c) $\overline{V_{j+1}} \subset V_j$.

(d) $\operatorname{diam}(V_{j+1}) \leq \frac{1}{2}\operatorname{diam}(V_j)$.

Since $\nu(X \times V_{k-1}) > 0$, then there is some $n_k > n_{k-1}$ so that $\nu((X \times V_{k-1}) \cap T_\Delta^{-n_k}(V_{k-1} \times X)) > 0$, thus we can pick $(a, z_1, z_2) \in \operatorname{supp}(\nu) \cap (X \times V_{k-1}) \cap T_\Delta^{-n_k}(V_{k-1} \times X)$. Define V_k to be an open ball around (a, z_1, z_2) with radius small enough so that $\operatorname{diam}(V_k) \leq \frac{1}{2}\operatorname{diam}(V_{k-1})$ and so that $\{a\} \times V_k \subset (X \times V_{k-1}) \cap T_\Delta^{-n_k}(V_{k-1} \times X)$. Then each condition laid out above is satisfied, either by definition or as an easy consequence of our choice of V_k. Thus by induction we can define such a sequence $\{X \times V_j\}_{j=1}^\infty$ as well as a sequence $\{n_j\}_{j=1}^\infty$ all satisfying $(a) - (d)$.

Consider now the intersection of the closures of all the V_j, $\bigcap_{j=0}^\infty \overline{V_j} \subset X^2$. Since $\operatorname{diam}(\overline{V_j}) = \operatorname{diam}(V_j)$, then $\operatorname{diam}(\overline{V_j}) \to 0$ as $j \to \infty$, and X compact implies each $\overline{V_j}$ is compact, thus $\bigcap_{j=0}^\infty \overline{V_j} = \{(x_1, x_2)\}$. We claim that (a, x_1, x_2) is our Erdős progression with the sequence $c(i) = n_i$. Observe that for each n_j, $T_\Delta^{n_j}(a, x_1) \in V_{j-1}$ from condition (a), and since $\operatorname{diam}(V_{j-1}) \to 0$, then $T^{n_j}(a, x_1) \to (x_1, x_2)$, thus we have an Erdős progression. ∎

We are then left with the task of proving that every ergodic system (X, μ, T) has a progressive measure supported on any of its quasi-generic points, and that if E is open and $\mu(E) > 0$, then the set $X \times X \times E$ or $X \times X \times T^{-1}E$ has positive ν-measure in order to satisfy Theorem 3.4. We will restate this as a theorem, after which we will proceed to define a candidate measure which we will spend the rest of the paper proving is progressive.

Theorem 4.3. *Let (X, μ, T) be an ergodic system, Φ a Følner sequence, $a \in \mathbf{gen}(\mu, \Phi)$, $E \subset X$ with $\mu(E) > 0$. Then there exists a progressive measure $\sigma \in \mathcal{M}(X^3)$ with $\sigma(\{a\} \times X^2) = 1$ and $\sigma(X \times X \times T^{-t}E) > 0$ for some $t \in \{0, 1\}$.*

Proof that Theorem 4.3 implies Theorem 3.4. By Lemma 4.2, if σ is progressive and $\sigma(\{a\} \times X^2) = 1$, $\sigma(X \times X \times T^{-t}E) > 0$, then there is an Erdős progression (a, x_1, x_2) with $x_2 \in T^{-t}E$. This is precisely what Theorem 3.4 demanded. ∎

Recall that every ergodic system has a largest factor that is isomorphic to rotation on a compact group, some useful facts about this factor are collected in the introduction.

Definition 4.4. Let (X, μ, T) be an ergodic system, Φ a Følner sequence, $a \in \mathbf{gen}(\mu, \Phi)$. We define the measure ξ on $Z_1 \times Z_1$ to be the following. Let $\pi_1 : X \to Z_1$ denote the factor map from X to its Kronecker factor. Then define

$$\xi \coloneqq \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \delta_{T^n(\pi_1(a))} \times \delta_{T^{2n}(\pi_1(a))}.$$

Proposition 4.5. *The measure ξ exists, is independent of Φ, and its marginals satisfy $\xi_1 = m_{Z_1}$, $\xi_2 = (R_{\pi_1(a)})_* m_{2Z_1}$.*

Proof. Let $Y \coloneqq \{(z, 2z) : z \in Z_1\} \leqslant Z_1^2$, we claim $\xi = (R_{(\pi_1(a), \pi_1(a))})_* m_Y$, where m_Y is Haar measure on Y. Since (X, μ, T) is ergodic, then (Z_1, m_{Z_1}, T) is ergodic. So if α is such that $T(z) = z + \alpha$, then $\{n\alpha : n \in \mathbb{Z}\}$ is dense in Z_1 (see the comments in the preliminary section about compact groups under rotation). It follows that $\{(n\alpha, 2n\alpha) : n \in \mathbb{Z}\}$ is dense in $Y = \{(z, 2z) : z \in Z_1\}$. Now ξ is a weak-* limit of

$$\xi_N = \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T^n \times T^{2n})_* \delta_{(\pi_1(a), \pi_1(a))},$$

$$(R_{(-\pi_1(a), -\pi_1(a))})_* \xi_N = \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T^n \times T^{2n})_* \delta_{(e,e)} \to m_Y = (R_{(-\pi_1(a), -\pi_1(a))})_* \xi,$$

the second convergence is due to both ([EW10], Theorem 4.1) and our comments about Kronecker systems in the preliminary section. $(R_{(-\pi_1(a),-\pi_1(a))})_*\xi_N$ will limit to a $T \times T^2$-invariant measure supported on Y, and our preliminary remarks imply that any measure invariant under rotation by an element with dense orbit must be the Haar measure. The uniqueness of this limit gives independence of the Følner set Φ. This yields what was desired, that $\xi = (R_{(\pi_1(a),\pi_1(a))})_* m_Y$.

From the definition of ξ, its first marginal can be computed as

$$\xi_1 = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \delta_{T^n \pi_1(a)} = m_{Z_1},$$

since the partial sums $\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} \delta_{T^n\pi_1(a)}$ converge to a T-invariant measure on Z_1, and by unique ergodicity this must be the Haar measure.

We have that its second marginal $\xi_2 = (R_{\pi_1(a)})_* m_{2Z_1}$, as

$$\xi_2 = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \delta_{T^{2n}\pi_1(a)},$$

and by the argument we laid out above, since $\{(2n)\alpha : n \in \mathbb{Z}\} + \pi_1(a)$ is dense in $2Z_1 + \pi_1(a)$, then $\xi_2 = (R_{\pi_1(a)})_* m_{2Z_1}$. ■

Definition 4.6. Let (X, μ, T) be an ergodic system, Φ Følner sequence, $a \in \mathbf{gen}(\mu, \Phi)$, ξ as in Definition 4.4. We will now define our candidate progressive measure. Define $\sigma \in \mathcal{M}(X^3)$ as follows. For $f_0, f_1, f_2 \in C(X)$, denote by $f_0 \otimes f_1 \otimes f_2$ the function $f_0(x_0)f_1(x_1)f_2(x_2)$ on X^3. Then define

$$\int_{X^3} f_0 \otimes f_1 \otimes f_2 d\sigma := f_0(a) \int_{Z_1^2} E(f_1|\mathscr{Z}_1) \otimes E(f_2|\mathscr{Z}_1) d\xi.$$

This candidate measure is progressive when the system (X, μ, T) has continuous factor map to its Kronecker factor, and the following proposition will allow us to reduce to that case.

Proposition 4.7 ([KMRR23], Proposition 3.20). *Let (X, μ, T) be an ergodic system, $a \in \mathbf{gen}(\mu, \Phi)$. Then there is an ergodic system $(\tilde{X}, \tilde{\mu}, \tilde{T})$, a Følner sequence Ψ, $\tilde{a} \in \mathbf{gen}(\tilde{\mu}, \Psi)$, and a continuous factor map $\pi : \tilde{X} \to X$ with $\pi(\tilde{a}) = a$ such that $(\tilde{X}, \tilde{\mu}, \tilde{T})$ has a continuous factor map to its Kronecker factor.*

We claim that for every ergodic system with continuous factor map to its Kronecker factor, and for every quasi-generic point a, σ is progressive and satisfies positivity of $X \times X \times T^{-t}E$.

Theorem 4.8. *Let (X, μ, T) be an ergodic system with continuous factor map to its Kronecker factor, Φ Følner sequence, $a \in \mathbf{gen}(\mu, \Phi)$, $E \subset X$ open with $\mu(E) > 0$. Then if σ is defined as above, $\sigma(X \times X \times T^{-t}E) > 0$ for some $t \in \{0, 1\}$.*

Proof. Since (X, μ, T) is ergodic, then its Kronecker factor (Z_1, m_{Z_1}, T) is also ergodic. If $\alpha \in Z_1$ is such that $T(z) = z + \alpha$, then $\{n\alpha : n \in \mathbb{Z}\}$ is dense in Z_1. We claim that the subgroup $2Z_1 = \{2z : z \in Z_1\} \leqslant Z_1$ has $[Z_1 : 2Z_1] = 1, 2$.

To prove the claim, suppose $2Z_1 \neq Z_1$, we will show that it has index two. We have that $\{(2n)\alpha : n \in \mathbb{Z}\}$ is dense in $2Z_1$ and that $\{(2n+1)\alpha : n \in \mathbb{Z}\}$ is dense in $2Z_1 + \alpha$. With this, since $\{(2n)\alpha : n \in \mathbb{Z}\} \cup \{(2n+1)\alpha : n \in \mathbb{Z}\} = \{n\alpha : n \in \mathbb{Z}\}$, and noting $\overline{Y} \cup \overline{X} = \overline{Y \cup X}$, then we see that $2Z_1 \cup (2Z_1 + \alpha) = Z_1$, so indeed, $[Z_1 : 2Z_1] = 2$.

In the case where $2Z_1 \neq Z_1$, notice that $m_{2Z_1}(A) = 2m_{Z_1}(A \cap 2Z_1)$, and so $T_* m_{2Z_1}(A) = m_{2Z_1}(A - \alpha) = 2m_{Z_1}((A - \alpha) \cap 2Z_1) = 2m_{Z_1}(A \cap (2Z_1 + \alpha))$, noting that $(A - \alpha) \cap 2Z_1 = A \cap (2Z_1 + \alpha) - \alpha$ and using shift-invariance of m_{Z_1}. Thus we have

$$m_{2Z_1} + T_* m_{2Z_1} = 2m_{Z_1},$$

as $(m_{2Z_1} + T_* m_{2Z_1})(A) = 2(m_{Z_1}(A \cap 2Z_1) + m_{Z_1}(A \cap (2Z_1 + \alpha)) = 2m_{Z_1}(A)$ since $2Z_1 \cup (2Z_1 + \alpha) = Z_1$. Recall that $\xi_2 = (R_{\pi_1(a)})_* m_{2Z_1}$, so we have that $\xi_2 + T_* \xi_2 = 2m_{Z_1}$.

Now if $E \subset X$ is open and $\mu(E) > 0$, then there is $f : X \to [0,1]$ continuous so that $\text{supp}(f) \subset E$ and $\int f d\mu > 0$, then

$$(\star) \qquad \sigma(X \times X \times E) \geq \int_{X^3} \mathbf{1} \otimes \mathbf{1} \otimes f d\sigma = 1 \cdot \int_{Z_1^2} \underbrace{E(\chi_X|\mathscr{Z}_1)(x)}_{=1 \text{ a.e.}} E(f|\mathscr{Z}_1)(y) d\xi(x,y) = \int_{Z_1} E(f|\mathscr{Z}_1)(y) d\xi_2(y).$$

So we have

$$\begin{aligned} 0 < \int_X f d\mu = \int_{Z_1} E(f|\mathscr{Z}_1)(y) dm_{Z_1}(y) &= \frac{1}{2}\left(\int_{Z_1} E(f|\mathscr{Z}_1)(y) d(\xi_2 + T_*\xi_2)\right) \\ &= \frac{1}{2}\left(\int_{Z_1} E(f|\mathscr{Z}_1)(y) d\xi_2 + \int_{Z_1} E(f|\mathscr{Z}_1)(T(y)) d\xi_2\right) \\ &= \frac{1}{2}\left(\int_{Z_1} E(f|\mathscr{Z}_1)(y) d\xi_2 + \int_{Z_1} E(f \circ T|\mathscr{Z}_1)(y) d\xi_2\right) \\ &\leq \frac{1}{2}\left(\sigma(X \times X \times E) + \sigma(X \times X \times T^{-1}E)\right), \end{aligned}$$

since $\text{supp}(f \circ T) \subset T^{-1}E$. So one of $\sigma(X \times X \times E), \sigma(X \times X \times T^{-1}E)$ must be positive, which is what we wanted.

If $2Z_1 = Z_1$, then $\xi_2 = (R_{\pi_1(a)})_* m_{Z_1} = m_{Z_1}$, so using $(\star)$,

$$\sigma(X \times X \times E) \geq \int_{Z_1} E(f|\mathscr{Z}_1)(y) d\xi_2(y) = \int_{Z_1} f dm_{Z_1} > 0.$$

So we have what was desired in all cases. ∎

Remark. The assumption of a dense cyclic subgroup is necessary for this fact about $2Z_1$ to be true. If $G = \prod_{i=1}^{\infty}(\mathbb{Z}/2\mathbb{Z})$, then every element $g \in G$ has $2g = e$, so $2G = \{e\}$ has infinite index in G, but G is missing a dense cyclic subgroup.

Remark. The proof of Theorem 4.8 shown here is not the original proof provided in [KMRR26], nor is it the original theorem statement. Thanks to a suggestion by Professor Bryna Kra, we were advised to investigate $X \times X \times T^{-t}E$, as in the original work, this theorem showed $\sigma(X \times T^{-t}E \times T^{-t}E) > 0$. This was necessary in their case, as the combinatorial goal of the paper was more delicate, but in our case it was only necessary to show $\sigma(X \times X \times T^{-t}E) > 0$. This simplified the proof of this theorem, as in the original work, the proof of positivity required use of the Furstenberg Joining and relating averages of σ to the joining, while in this thesis, the proof of positivity is elementary.

Theorem 4.9. *Let (X, μ, T) be an ergodic system with continuous factor map to its Kronecker factor, Φ Følner sequence, $a \in \mathbf{gen}(\mu, \Phi)$, $E \subset X$ open with $\mu(E) > 0$. Then if σ is defined as above, σ is progressive.*

We will prove this theorem in the next section. With all three of these theorems, we can finally prove Theorem 4.3.

Proof that Proposition 4.7 + Theorem 4.8 + Theorem 4.9 imply Theorem 4.3. Let (X, μ, T) be an ergodic system, Φ a Følner sequence, $a \in \mathbf{gen}(\mu, \Phi)$, $E \subset X$ with $\mu(E) > 0$. Let $(\tilde{X}, \tilde{\mu}, \tilde{T})$ be an ergodic system with continuous factor map to its Kronecker factor, Ψ a Følner sequence, $\tilde{a} \in \mathbf{gen}(\tilde{\mu}, \Psi)$, $\pi : \tilde{X} \to X$ a continuous factor map with $\pi(\tilde{a}) = a$, all of which exist by Proposition 4.7. Then there is a progressive measure $\tilde{\sigma} \in \mathcal{M}(\tilde{X}^3)$ with $\tilde{\sigma}(\{\tilde{a}\} \times \tilde{X}^2) = 1$, and since $E \subset X$ is open, then $\pi^{-1}E$ is open in $\tilde{X}$. Additionally, $\tilde{\mu}(\pi^{-1}E) = \mu(E) > 0$, so there is some $t \in \{0, 1\}$ with $\tilde{\sigma}(X \times X \times \tilde{T}^{-t}(\pi^{-1}E)) > 0$.

Define $\pi^3 : \tilde{X}^3 \to X^3$ by $\pi^3(x_1, x_2, x_3) = (\pi(x_1), \pi(x_2), \pi(x_3))$, $\sigma := (\pi^3)_*\tilde{\sigma}$. Note that $\sigma(\{a\} \times X^2) = \tilde{\sigma}(\pi^{-1}\{a\} \times \tilde{X}^2) = 1$ since $\pi(\tilde{a}) = a$, and $\sigma(X \times X \times T^{-t}E) = \tilde{\sigma}(\tilde{X} \times \tilde{X} \times \pi^{-1}(T^{-t}E)) = \tilde{\sigma}(\tilde{X} \times \tilde{X} \times \tilde{T}^{-t}(\pi^{-1}E)) > 0$. We claim $\sigma \in \mathcal{M}(X^3)$ is a progressive measure on X, once we have proved that, then we will have Theorem 4.3. If $\sigma(X \times U_1 \times U_2) > 0$, then $\tilde{\sigma}(\tilde{X} \times \pi^{-1}U_1 \times \pi^{-1}U_2) > 0$, so there are infinitely many n with $\tilde{\sigma}((\tilde{X} \times \pi^{-1}U_1 \times \pi^{-1}U_2) \cap \tilde{T}_{\Delta}^{-n}(\pi^{-1}U_1 \times \pi^{-1}U_2 \times \tilde{X})) > 0$, and $\tilde{\sigma}((\tilde{X} \times \pi^{-1}U_1 \times \pi^{-1}U_2) \cap \tilde{T}_{\Delta}^{-n}(\pi^{-1}U_1 \times \pi^{-1}U_2 \times \tilde{X})) = \sigma((X \times U_1 \times U_2) \cap T_{\Delta}^{-n}(U_1 \times U_2 \times X))$, so indeed, σ is progressive, and thus we have Theorem 4.3. ∎

Before we prove that σ is progressive, we collect and prove two useful facts.

Lemma 4.10. *σ is $Id \times T \times T^2$-invariant, and its marginals $\sigma_i = (\pi_i)_*\sigma$ for $i = 0, 1, 2$, where $\pi_i : X^3 \to X$ is $\pi_i(x_0, x_1, x_2) = x_i$, satisfy*

$$\sigma_0 = \delta_a, \sigma_1 \leq \mu, \sigma_2 \leq 2\mu.$$

Proof. It suffices to check that $\int g \circ (Id \times T \times T^2) d\sigma = \int g d\sigma$ for each $g \in C(X^3)$, using Stone-Weierstrass we check this for functions of the form $f_0 \otimes f_1 \otimes f_2$ with $f_i \in C(X)$. Let $f_0 \otimes f_1 \otimes f_2 \in C(X^3)$, then

$$\begin{aligned}
\int_{X^3} (f_0 \otimes f_1 \otimes f_2) \circ Id \times T \times T^2 d\sigma &= f_0(a) \int_{Z_1^2} E(f_1 \circ T|\mathscr{Z}_1) \otimes E(f_2 \circ T^2|\mathscr{Z}_1) d\xi \\
&= f_0(a) \int_{Z_1^2} E(f_1|\mathscr{Z}_1) \circ T \otimes E(f_2|\mathscr{Z}_1) \circ T^2 d\xi \\
\xi \text{ is } T \times T^2\text{-invariant} &= f_0(a) \int_{Z_1^2} E(f_1|\mathscr{Z}_1) \otimes E(f_2|\mathscr{Z}_1) d\xi \\
&= \int_{X^3} f_0 \otimes f_1 \otimes f_2 d\sigma.
\end{aligned}$$

For the marginals of σ, that $\sigma_0 = \delta_a$ is obvious, for the others, using Proposition 4.5,

$$\int \mathbf{1} \otimes f \otimes \mathbf{1} d\sigma = \int E(f|\mathscr{Z}_1) \otimes \mathbf{1} d\xi = \int E(f|\mathscr{Z}_1) d\xi_1 = \int E(f|\mathscr{Z}_1) dm_{Z_1} = \int f d\mu,$$
$$\int \mathbf{1} \otimes \mathbf{1} \otimes f d\sigma = \int \mathbf{1} \otimes E(f|\mathscr{Z}_1) d\xi = \int E(f|\mathscr{Z}_1) d\xi_2 \leq 2 \int E(f|\mathscr{Z}_1) dm_{Z_1} = 2 \int f d\mu,$$

since $\xi_2 = (R_{\pi_1(a)})_* m_{2Z_1} \leq 2m_{Z_1}$. ■

Before embarking on the proof of progressiveness, we summarize the reduction from the original problem to proving progressiveness of σ. Once we have shown that for any $a \in \mathbf{gen}(\mu, \Phi)$, the corresponding measure σ is progressive, then we will know that for any open set $E \subset X$ with $\mu(E) > 0$, there will exist an Erdős progression (a, x_1, x_2) with $x_2 \in E$. Erdős progressions, in turn, give rise to sumsets in the return times of a to open neighborhoods of x_2, namely E. Then by the correspondence principle, we know that any positive density set can be encoded as return times of a point a to E. Writing out the implications in a chain, the reduction can be laid out as follows:

Proposition 4.7 + *Theorem* 4.8 + *Theorem* 4.9 $\implies$ *Theorem* 4.3 $\overset{\text{via Lemma 4.2}}{\implies}$ *Theorem* 3.4 $\overset{\text{ergodic reduction}}{\implies}$ *Theorem* 3.2 $\overset{\text{via Lemma 3.3}}{\implies}$ *Theorem* 1.2.

All that remains is to prove Theorem 4.9, which requires the most work out of the theorems we have proven so far.

5. Proving σ is progressive

Theorem 5.1. *Let (X, T) be a topological dynamical system, Φ a Følner sequence, and $\nu \in \mathcal{M}(X^2)$ be $T \times T^2$-invariant. If $G : X^2 \to [0, 1]$ is continuous and $\int_{X^2} G d\nu > 0$, then*

$$\liminf_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \int_{X^2} G \cdot (T_\Delta^n G) d\nu > 0. \tag{5.1}$$

Theorem 5.2. *Let (X, μ, T) be an ergodic m.p.s with a continuous factor map to its Kronecker factor, Φ a Følner sequence, and $a \in \mathbf{gen}(\mu, \Phi)$. Then for any continuous $G : X^2 \to \mathbb{C}$, we have:*

$$\lim_{N \to \infty} \left\| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} (T_\Delta^n (G \otimes 1) - T_\Delta^n (1 \otimes G)) \right\|_{L^2(\sigma)} = 0. \tag{5.2}$$

Before proving either of these theorems, we will show that they do, in fact, prove Theorem 4.9.

Proof that Theorem 5.1 + Theorem 5.2 imply Theorem 4.9. Let U_1, U_2 be open in X, and suppose $\sigma(X \times U_1 \times U_2) > 0$. We wish to show that for infinitely many n, $\sigma((X \times U_1 \times U_2) \cap T_\Delta^{-n}(U_1 \times U_2 \times X)) > 0$. Let $f_1, f_2 \in C(X)$ be such that $\operatorname{supp}(f_i) \subset U_i$, $0 \le f_i \le 1$, and $\int_{X^3}(\mathbf{1}\otimes f_1 \otimes f_2)d\sigma > 0$. Now by applying Theorem 5.1 to $\tilde{\sigma} = \psi_*\sigma$, where $\psi : X^3 \to X^2$ is given by $\psi(x_1, x_2, x_3) = (x_2, x_3)$, we see that since $\tilde{\sigma}$ is $(T \times T^2)$-invariant,

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^3} (\mathbf{1}\otimes f_1 \otimes f_2)\cdot T_\Delta^n(\mathbf{1}\otimes f_1 \otimes f_2)d\sigma = \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^2} (f_1 \otimes f_2)\cdot T_\Delta^n(f_1 \otimes f_2)d\tilde{\sigma} > 0.$$

Furthermore, we claim that by applying Theorem 5.2, we have

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^3} (\mathbf{1}\otimes f_1 \otimes f_2)\cdot T_\Delta^n(\mathbf{1}\otimes f_1 \otimes f_2)d\sigma = \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^3} (\mathbf{1}\otimes f_1 \otimes f_2)\cdot T_\Delta^n(f_1 \otimes f_2 \otimes \mathbf{1})d\sigma.$$

This can be seen by applying the Cauchy-Schwarz inequality to the difference of the terms of each sequences, followed by taking the limit inferior.

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^3} (\mathbf{1}\otimes f_1 \otimes f_2)\cdot (T_\Delta^n(\mathbf{1}\otimes f_1 \otimes f_2) - T_\Delta^n(f_1 \otimes f_2 \otimes \mathbf{1}))d\sigma \le \\
&\liminf_{N\to\infty} \|\mathbf{1}\otimes f_1 \otimes f_2\|_{L^2(\sigma)} \cdot \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T_\Delta^n(\mathbf{1}\otimes f_1 \otimes f_2) - T_\Delta^n(f_1 \otimes f_2 \otimes \mathbf{1})) \right\|_{L^2(\sigma)} \overset{(5.2)}{=} 0.
\end{aligned}$$

The same computation with $\mathbf{1}\otimes f_1 \otimes f_2$ and $f_1 \otimes f_2 \otimes \mathbf{1}$ interchanged proves the claim above. Altogether, we have the following string of inequalities,

$$\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \sigma((X \times U_1 \times U_2) \cap T_\Delta^{-n}(U_1 \times U_2 \times X)) \\
&\ge \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^3} (\mathbf{1}\otimes f_1 \otimes f_2)\cdot T_\Delta^n(f_1 \otimes f_2 \otimes \mathbf{1})d\sigma \\
&= \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^3} (\mathbf{1}\otimes f_1 \otimes f_2)\cdot T_\Delta^n(\mathbf{1}\otimes f_1 \otimes f_2)d\sigma > 0.
\end{aligned}$$

Then we necessarily have infinitely many n so that $\sigma((X \times U_1 \times U_2) \cap T_\Delta^{-n}(U_1 \times U_2 \times X) > 0$, proving that σ is progressive. ∎

We first prove Theorem 5.1 via the following lemma which only requires Roth's theorem, Theorem 2.1.

Lemma 5.3. *Let $\nu \in \mathcal{M}(X^2)$ be a $T \times T^2$-invariant measure. If $A_1, A_2 \subset X$ are such that $\nu(A_1 \times A_2) > 0$, defining $A := A_1 \times A_2$, then for any Følner sequence $(\Phi_N) = \Phi$, we have that*

(5.3)
$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \nu(A \cap T_\Delta^{-n} A) > 0.$$

Proof. Define $v := (1, 2), w := (1, 2)$, and for $u = (u_1, u_2) \in \mathbb{N}^2$, define $T_u := T^{u_1} \times T^{u_2}$. Then

$$\begin{aligned}
T_v^{-2}(A_1 \times X) &= T^{-2}A_1 \times X = T_\Delta^{-2}(A_1 \times X), \\
T_v^{-1}(X \times A_2) &= X \times T^{-2}A_2 = T_\Delta^{-2}(X \times A_2).
\end{aligned}$$

So for any n,

(5.3.1)
$$T_\Delta^{-2n}(A) = T_\Delta^{-2n}(A_1 \times X) \cap T_\Delta^{-2n}(X \times A_2) = T_v^{-2n}(A_1 \times X) \cap T_v^{-n}(X \times A_2) \supset T_v^{-2n}A \cap T_v^{-n}A.$$

After this manipulation, one can begin to see how Roth's Theorem (Theorem 2.1) will help in proving this lemma. First, define $\Psi_N := \{n \in \mathbb{N} : 2n \in \Phi_N\}$, $E_N := \Phi_N \cap 2\mathbb{N}, O_N := \Phi_N \cap (2\mathbb{N} + 1)$, we will use the fact that

$$\frac{|\Psi_N|}{|\Phi_N|} = \frac{|E_N|}{|\Phi_N|} \to \frac{1}{2}.$$

The proof of this fact can be found in the appendix, Lemma A.1.

We now have

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \nu(A\cap T_\Delta^{-n}A) &\geq \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N\cap 2\mathbb{N}} \nu(A\cap (T_\Delta^{-n}A)) \\
&= \underbrace{\lim_{N\to\infty} \left(\frac{|\Phi_N|}{2|\Psi_N|}\right)}_{=1} \liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N\cap 2\mathbb{N}} \nu(A\cap (T_\Delta^{-n}A)) \\
&= \liminf_{N\to\infty} \frac{1}{2|\Psi_N|} \sum_{n\in\Psi_N} \nu(A\cap (T_\Delta^{-2n}A)) \\
&\overset{(5.3.1)}{\geq} \liminf_{N\to\infty} \frac{1}{2|\Psi_N|} \sum_{n\in\Psi_N} \nu(A\cap (T_v^{-2n}A\cap T_v^{-n}A)).
\end{aligned}$$

Roth's theorem guarantees positivity of the limit inferior on the right by applying the theorem to the system $(X^2, T\times T^2, \nu)$, giving the desired result. ■

Theorem 5.1 is now at hand, all that remains to be done is approximate integrals of continuous functions G with the measure of open sets contained in the support of G, allowing us to apply Lemma 5.3.

Proof of Theorem 5.1. Since G is a continuous function from X^2 to $[0,1]$ with positive ν-integral, it follows that there are open sets U_1, U_2 and a constant $c>0$ so that $c(\chi_{U_1}\otimes\chi_{U_2})\leq G$ and $\nu(U_1\times U_2)>0$. Apply the previous lemma to these open sets U_1, U_2 and let $U = U_1\times U_2$ to obtain

$$\liminf_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \nu(U\cap T_\Delta^{-n}U) > 0. \tag{5.1.1}$$

Since $c(\chi_{U_1}\otimes\chi_{U_2}) = c\chi_U \leq G$, then for each N we have

$$\frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \int_{X^2} G\cdot(G\circ T_\Delta^n)d\nu \geq \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} c\int_{X^2} \chi_U\cdot\chi_{T_\Delta^{-n}U}d\nu \geq \frac{c}{|\Phi_N|} \sum_{n\in\Phi_N} \nu(U\cap T_\Delta^{-n}U).$$

Taking the limit inferior, we obtain the desired result by applying (5.1.1). ■

To prove Theorem 5.2 we first prove a statement concerning averages of vectors in L^2 spaces.

Lemma 5.4. *Let (X,μ) be a probability space, $\Phi\subset\mathbb{N}$ a finite set, $\{f_n\}_{n\in\Phi}\subset L^2(\mu)$, and finally $b:\Phi\to\mathbb{C}$. Then we have the following:*

$$\left\|\frac{1}{|\Phi|}\sum_{n\in\Phi} b(n)f_n\right\|_{L^2(\mu)}^2 \leq \|b\|_\infty^2 \left\|\frac{1}{|\Phi|}\sum_{n\in\Phi} f_n\otimes\overline{f_n}\right\|_{L^2(\mu\times\mu)}. \tag{5.4}$$

Proof. First we have

$$\left\|\frac{1}{|\Phi|}\sum_{n\in\Phi} b(n)f_n\right\|_{L^2(\mu)}^2 = \int \left(\frac{1}{|\Phi|}\sum_{n\in\Phi} f_n\right)\overline{\left(\frac{1}{|\Phi|}\sum_{m\in\Phi} f_m\right)}d\mu = \frac{1}{|\Phi|^2}\sum_{n,m\in\Phi}\int f_n\overline{f_m}d\mu.$$

Applying the Cauchy-Schwarz inequality and consolidating, we get

$$
\begin{aligned}
\left(\sum_{n,m\in\Phi}\left|\int f_n\overline{f_m}d\mu\right|\cdot 1\right)^2 &\leq \left(\sum_{n,m\in\Phi}\left|\int f_n\overline{f_m}d\mu\right|^2\right)\left(\sum_{n,m\in\Phi}1\right)\\
&= |\Phi|^2\sum_{n,m\in\Phi}\int_{X\times X}(f_n\otimes\overline{f_n})\cdot(\overline{f_m}\otimes f_m)d(\mu\times\mu)\\
&\leq |\Phi|^2\int_{X\times X}\left|\sum_{n,m\in\Phi}(f_n\otimes\overline{f_n})\cdot(\overline{f_m}\otimes f_m)\right|d(\mu\times\mu)\\
&= |\Phi|^2\int_{X\times X}\left|\sum_{n\in\Phi}(f_n\otimes\overline{f_n})\right|^2 d(\mu\times\mu)\\
&= |\Phi|^2\left\|\sum_{n\in\Phi}f_n\otimes\overline{f_n}\right\|^2_{L^2(\mu\times\mu)}.
\end{aligned}
$$

Scaling the above inequality we derived by $\frac{1}{|\Phi|^4}$, we obtain

$$
\left(\frac{1}{|\Phi|^2}\sum_{n,m\in\Phi}\left|\int f_n\overline{f_m}d\mu\right|\cdot 1\right)^2 \leq \left\|\frac{1}{|\Phi|}\sum_{n\in\Phi}f_n\otimes\overline{f_n}\right\|^2_{L^2(\mu\times\mu)}.
$$

Finally, combining the first computation with this inequality, we get what was desired, namely

$$
\left\|\frac{1}{|\Phi|}\sum_{n\in\Phi}b(n)f_n\right\|^2_{L^2(\mu)} \leq \frac{1}{|\Phi|^2}\sum_{n,m\in\Phi}\int f_n\overline{f_m}d\mu \leq \frac{1}{|\Phi|^2}\sum_{n,m\in\Phi}\left|\int f_n\overline{f_m}d\mu\right| \leq \left\|\frac{1}{|\Phi|}\sum_{n\in\Phi}f_n\otimes\overline{f_n}\right\|_{L^2(\mu\times\mu)}.
$$

■

To complete the proof of Theorem 5.2, we must first introduce the concept of uniform seminorms. We will only be using the first and second uniform seminorms. The proof of their existence and some of their properties will not be provided here, but they are central to the proof of progressiveness.

Definition 5.5. Let (X,μ,T) be a measure preserving system. Define for any $f\in L^\infty(X,\mu)$ the s-step uniform seminorm for $s\in\{0,1,2\}$:

$$
\begin{aligned}
\|f\|_{U^0(X,\mu,T)} &= \int_X f\ d\mu,\\
\|f\|^2_{U^1(X,\mu,T)} &= \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\|T^hf\cdot\overline{f}\|_{U^0(X,\mu,T)} = \|E(f|\mathscr{E})\|^2_{L^2(\mu)},\\
\|f\|^4_{U^2(X,\mu,T)} &= \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\|T^hf\cdot\overline{f}\|^2_{U^1(X,\mu,T)}.
\end{aligned}
$$

Let Z_1 denote the Kronecker factor of (X,μ,T), then if (X,μ,T) is ergodic,

$$
E(f|\mathscr{Z}_1)=0 \iff \|f\|_{U^2(X,\mu,T)}=0.
$$

We show below the analagous statement is true for the case $k=1$ with $E(f|\mathscr{E})$, but we use the statement for $U^2(X,\mu,T)$ without proof.

Remark. The uniform seminorms are defined inductively for any $s\in\mathbb{N}$ as

$$
\|f\|^{2^{s+1}}_{U^{s+1}(X,\mu,T)} \coloneqq \lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\|T^hf\cdot\overline{f}\|^{2^s}_{U^s(X,\mu,T)}.
$$

The proof of their existence and further discussion can be found in [HK18]. These seminorms play a crucial role in the structure theory of ergodic systems. Of particular interest is the fact that these seminorms control averages of the form

$$\left\| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k \right\|_{L^2(\mu)}$$

in much the same way as Lemma 5.6 part (b). We refer the reader to Kra's survey [Kra06], specifically Sections 5 and 6.

Computation for existence of $\|f\|_{U^1(X,\mu,T)}$.

$$\|f\|_{U^1(X,\mu,T)}^2 = \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \int_X f \circ T^h \cdot \overline{f}\, d\mu = \lim_{H\to\infty} \left\langle \frac{1}{H} \sum_{h=1}^{H} f \circ T^h, f \right\rangle_{L^2(\mu)} \stackrel{\star}{=} \langle E(f|\mathscr{E}), f\rangle_{L^2(\mu)},$$

where the equality marked by $\star$ is a consequence of the mean ergodic theorem. For any $\mathscr{A} \subset \mathscr{B}$, we have that $\langle E(f|\mathscr{A}), f\rangle = \|E(f|\mathscr{A})\|_{L^2(\mu)}^2$, as

$$\int_X g\, d\mu = \int_X E(g|\mathscr{A})\, d\mu,$$

giving

$$\int E(f|\mathscr{A})\overline{f}\, d\mu = \int E(E(f|\mathscr{A})\overline{f}|\mathscr{A})\, d\mu = \int E(f|\mathscr{A})E(\overline{f}|\mathscr{A})\, d\mu = \|E(f|\mathscr{A})\|_{L^2(\mu)}^2.$$

So $\|f\|_{U^1(X,\mu,T)}^2 = \|E(f|\mathscr{E})\|_{L^2(\mu)}^2$. ■

With the seminorms defined, we collect some useful properties of theirs to assist in the proof of Theorem 5.2.

Lemma 5.6. *Let (X,μ,T) be a measure preserving system, $f \in L^\infty(\mu)$, $\|f\|_{U^i(X,\mu,T)}$ for $i = 0,1,2$ defined as in Definition 5.5. Then*

(a) $\|f \otimes \overline{f}\|_{U^1(X,\mu,T)} \leq \|f\|_{U^2(X,\mu,T)}^2$.

(b) *For integers $c_1, c_2 \geq 1$, there is a constant C independent of the system so that for any $f_1, f_2 \in L^\infty(\mu)$ with $\|f_i\|_\infty \leq 1$ and for any Følner sequence Φ_N, we have*

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} T^{c_1 n} f_1 \cdot T^{c_2 n} f_2 \right\|_{L^2(\mu)} \leq C \min_{i=1,2} \{\|f_i\|_{U^2(X,\mu,T)}\}.$$

(c) *Let $c \in \mathbb{N}$, $\rho \in \mathcal{M}(X)$ be T^c-invariant, and $\rho \leq D\mu$ for some $D > 0$. Then for any $f \in L^\infty(\mu)$, we have*

$$\|f\|_{U^2(X,\rho,T^c)} \leq D\sqrt{c}\|f\|_{U^2(X,\mu,T)}.$$

Proof of Lemma 5.6 parts (a), (c). This is the same proof provided in ([FH16], Appendix A.1). Notice that

$$\left|\int f d\mu\right| = \left|\int E(f|\mathscr{A}) d\mu\right| \leq \int |E(f|\mathscr{A})|\, d\mu \leq \|E(f|\mathscr{A})\|_{L^2(\mu)} \cdot \|\mathbf{1}\|_{L^2(\mu)} = \|E(f|\mathscr{A})\|_{L^2(\mu)}$$

for any sub σ-algebra $\mathscr{A}$ of $\mathscr{B}$. In the case of $\mathscr{A} = \mathscr{E}$, this computation yields $\left|\int f d\mu\right| \leq \|E(f|\mathscr{E})\|_{L^2(\mu)} = \|f\|_{U^1(X,\mu,T)}$.

Now write $\mathscr{F}$ for the σ-algebra of $(T\times T)$-invariant sets in $X\times X$, $\mathscr{E}$ for the σ-algebra of T-invariant sets in X.

$$\begin{aligned}
\|f\otimes\overline{f}\|^2_{U^1(X\times X,\mu\times\mu,T\times T)} &= \|E(f\otimes\overline{f}|\mathscr{F})\|^2_{L^2(\mu\times\mu)} = \langle E(f\otimes\overline{f}|\mathscr{F}), f\otimes\overline{f}\rangle_{L^2(\mu\times\mu)} \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\langle T^n_\Delta(f\otimes\overline{f}), f\otimes\overline{f}\rangle \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\int(T^nf\cdot\overline{f})(x)d\mu(x)\int\overline{(T^nf\cdot\overline{f})}(y)d\mu(y) \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left|\int T^nf\cdot\overline{f}d\mu\right|^2 \\
&\leq \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\|T^nf\cdot\overline{f}\|^2_{U^1(X,\mu,T)} = \|f\|^4_{U^2(X,\mu,T)}.
\end{aligned}$$

We also have that for any $k\geq 0, c\geq 1$ that $\|f\|_{U^k(X,\mu,T^c)}\leq c^{\frac{k}{2^k}}\|f\|_{U^k(X,\mu,T)}$. We prove this by induction, the case $k=0$ is trivial since all quantities considered are the same. Now suppose we have the inequality for $k-1$, then

$$\|f\|^{2^k}_{U^k(X,\mu,T^c)} = \lim_{H\to\infty}\frac{1}{H}\sum_{h=0}^{H-1}\|T^{ch}f\cdot\overline{f}\|^{2^{k-1}}_{U^{k-1}(X,\mu,T^c)} \leq \lim_{H\to\infty}\frac{1}{H}\sum_{h=0}^{H-1}c^{\frac{k-1}{2^{k-1}}\cdot 2^{k-1}}\|T^{ch}f\cdot\overline{f}\|^{2^{k-1}}_{U^{k-1}(X,\mu,T)} \leq$$

$$\lim_{H\to\infty}\frac{c}{c}\cdot\frac{1}{H}\sum_{h=0}^{cH-1}c^{k-1}\|T^hf\cdot\overline{f}\|^{2^{k-1}}_{U^{k-1}(X,\mu,T)} = c^k\|f\|^{2^k}_{U^k(X,\mu,T)}.$$

$$\Rightarrow \|f\|_{U^k(X,\mu,T^c)}\leq c^{\frac{k}{2^k}}\|f\|_{U^k(X,\mu,T)}$$

Now examining (c), since $\rho\leq D\mu$, then $\|f\|_{U^2(X,\rho,T^c)}\leq D\|f\|_{U^2(X,\mu,T^c)}\leq Dc^{\frac{2}{4}}\|f\|_{U^2(X,\mu,T)}$.

The other item (b) is proved in [FK24], [Hos08], specifically (b) is a consequence of ([FK24], Lemma 3.1) with ([Hos08], Proposition 1). ■

While we provided citations for the proof of (b) above, we did attempt to recreate the proof independently over Følner sets. These attempts were unsuccessful, but we refer the reader to Section 5 of [Kra06] for insight into how the uniform seminorms control the average in (b) via the Van der Corput trick. We are now ready to prove two major ingredients in Theorem 5.2.

Lemma 5.7. *Let (X,μ,T) be a measure preserving system, $k\in\mathbb{N}$, and $\nu\in\mathcal{M}(X^2)$ be $(T\times T^2)$-invariant. Assume that the marginals of ν satisfy $\nu_i\leq i\mu$ for $i=1,2$. Then for $f_1,f_2\in L^\infty(X)$,*

$$\limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}T^n_\Delta(f_1\otimes f_2)\right\|_{L^2(\nu)}\leq 2^{\frac{3}{2}}C\min_{i=1,2}\left\{\|f_i\|_{U^2(X,\mu,T)}\right\}, \tag{5.7}$$

where C is as in Lemma 5.6 part (b).

Proof. Define Ψ_N as (5.7) in the proof of Lemma 5.3, that is $\Psi_N := \{n\in\mathbb{N} : 2n\in\Phi_N\} = \frac{1}{2}(\Phi_N\cap 2\mathbb{N})$. Similarly define an analogous set for odd elements, $\Theta_N := \{n\in\mathbb{N} : 2n+1\in\Phi_N\}$. Both of these sequences Ψ_N,Θ_N form Følner sequences in $\mathbb{N}$, this is proved in the appendix, Lemma A.2. If we define $E_N=\Phi_N\cap 2\mathbb{N}, O_N=\Phi_N\cap(2\mathbb{N}+1)$, then we see that $|\Psi_N|=|E_N|, |\Theta_N|=|O_N|$. By Lemma A.1 and its proof, we know that

$$\lim_{N\to\infty}\frac{|E_N|}{|\Phi_N|} = \lim_{N\to\infty}\frac{|O_N|}{|\Phi_N|} = \frac{1}{2},$$

so it follows that

$$\lim_{N\to\infty}\frac{|\Phi_N|}{2|\Psi_N|} = \lim_{N\to\infty}\frac{|\Phi_N|}{2|\Theta_N|} = 1. \tag{5.7.1}$$

Now for each N, we have

$$\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} T_\Delta^n(f_1\otimes f_2) = \frac{1}{|\Phi_N|}\left(\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2) + \sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right),$$

$$\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} T_\Delta^n(f_1\otimes f_2)\right\|_{L^2(\sigma)} \leq \left\|\frac{1}{|\Phi_N|}\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2)\right\|_{L^2(\sigma)} + \left\|\frac{1}{|\Phi_N|}\sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right\|_{L^2(\sigma)}.$$

Applying subadditivity of the limit superior and (5.7.1),

$$\begin{aligned}
(5.7.2)\qquad & \limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} T_\Delta^n(f_1\otimes f_2)\right\|_{L^2(\sigma)} \leq \\
& \limsup_{N\to\infty}\left(\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2)\right\|_{L^2(\sigma)} + \left\|\frac{1}{|\Phi_N|}\sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right\|_{L^2(\sigma)}\right) \leq \\
& \limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2)\right\|_{L^2(\sigma)} + \limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right\|_{L^2(\sigma)} = \\
& \limsup_{N\to\infty}\left\|\frac{1}{2|\Psi_N|}\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2)\right\|_{L^2(\sigma)} + \limsup_{N\to\infty}\left\|\frac{1}{2|\Theta_N|}\sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right\|_{L^2(\sigma)} \leq \\
& \frac{1}{2}\left(\limsup_{N\to\infty}\left\|\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2)\right\|_{L^2(\sigma)} + \limsup_{N\to\infty}\left\|\frac{1}{|\Theta_N|}\sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right\|_{L^2(\sigma)}\right).
\end{aligned}$$

So it suffices to show that (5.7) holds for each average along Ψ_N, Θ_N. Let $j\in\{0,1\}$, and define $g_{1,j}(x_1,x_2) := T^j f_1(x_1)\otimes \mathbf{1}$, $g_{2,j}(x_1,x_2) := \mathbf{1}\otimes T^j f_2(x_2)$. Let $S = T\times T^2$, and observe that

$$S^{2n} g_{1,j}(x_1,x_2) = T^{2n+j} f_1(x_1)\otimes \mathbf{1}, S^n g_{2,j}(x_1,x_2) = \mathbf{1}\otimes T^{2n+j} f_2(x_2).$$

With this reformulation, we can apply Lemma 5.6 part (b) to (X^2,ν,S), the functions $g_{i,j}$ and the Følner sequences Ψ_N if $j=0$, Θ_N if $j=1$ with $c_1=2, c_1=1$ to obtain

(5.7.3)

$$\limsup_{N\to\infty}\left\|\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N} T_\Delta^{2n}(f_1\otimes f_2)\right\|_{L^2(\nu)} = \limsup_{N\to\infty}\left\|\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N} S^{2n}g_{1,0}S^n g_{2,0}\right\|_{L^2(\nu)} \leq C\min_{i=1,2}\{\|g_{i,0}\|_{U^2(X^2,\nu,S)}\},$$

$$\limsup_{N\to\infty}\left\|\frac{1}{|\Theta_N|}\sum_{n\in\Theta_N} T_\Delta^{2n+1}(f_1\otimes f_2)\right\|_{L^2(\nu)} = \limsup_{N\to\infty}\left\|\frac{1}{|\Theta_N|}\sum_{n\in\Theta_N} S^{2n}g_{1,1}S^n g_{2,1}\right\|_{L^2(\nu)} \leq C\min_{i=1,2}\{\|g_{i,1}\|_{U^2(X^2,\nu,S)}\},$$

where $C>0$ is as in Lemma 5.6 part (b). Now $\|g_{i,j}\|_{U^2(X^2,\nu,S)} = \|T^j f_i\|_{U^2(X,\nu_i,T^i)}$ by examining the definition of $\|f\|_{U^2(X,\mu,T)}$, so we can apply Lemma 5.6 part (c) with $\rho=\nu_i$ and see that

$$\|g_{i,j}\|_{U^2(X^2,\nu,S)} = \|T^j f_i\|_{U^2(X,\nu_i,T^i)} \leq i^{\frac{3}{2}}\|T^j f_i\|_{U^2(X,\mu,T)} = i^{\frac{3}{2}}\|f_i\|_{U^2(X,\mu,T)},$$

since $\|f\|_{U^2(X,\mu,T)} = \|T^k f\|_{U^2(X,\mu,T)}$ for any k by definition. After combining this observation with (5.7.2) and (5.7.3), we have

$$\limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} T_\Delta^n(f_1\otimes f_2)\right\|_{L^2(\sigma)} \leq 2^{\frac{3}{2}}C\min_{i=1,2}\{\|f_i\|_{U^2(X,\mu,T)}\}.$$

■

Lemma 5.8. *Let (X,μ,T) be an ergodic system, $\Phi=(\Phi_N)_N$ Følner, and left $\tau\in\mathcal{M}(X^3)$ be $Id\times T\times T^2$- invariant. Assume further that the marginals τ_i have $\tau_i\leq i\mu$ for $i=1,2$, then for any $f\in L^\infty(\mu)$ and any bounded $b:\mathbb{N}\to\mathbb{C}$, we have*

$$\limsup_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} b(n)\cdot(\mathbf{1}\otimes T^n f\otimes\mathbf{1})\right\|_{L^2(\tau)} \leq \|b\|_\infty\|f\|_{U^2(X,\mu,T)}. \tag{5.8}$$

Proof. Let $\nu = \tau_1 \times \tau_1$, where $\tau_1 = (\pi_1)_*(\tau)$, $\pi_1 : X^3 \to X$ by $\pi_1(x_0, x_1, x_2) = x_1$, or equivalently $\nu = (\pi_1 \times \pi_1)_*(\tau \times \tau)$. Then ν is $T \times T$-invariant, and $\tau_1 \leq \mu$, but then $\tau_1 = \mu$ since they are both probability measures, as if $\tau_1(A) < \mu(A)$, then $\tau_1(A^c) > \mu(A^c)$, a contradiction. So $\nu = \tau_1 \times \tau_1 = \mu \times \mu$.

By the definition of the uniform seminorms, we see that

$$\limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T \times T)^n (f \otimes \overline{f}) \right\|_{L^2(\mu\times\mu)} = \|f \otimes \overline{f}\|_{U^1(X\times X, \mu\times\mu, T\times T)}.$$

Now by Lemma 5.6 part (a), we have that $\|f \otimes \overline{f}\|_{U^1(X\times X,\mu\times\mu,T\times T)} \leq \|f\|^2_{U^2(X,\mu,T)}$, so in summary we obtain

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T \times T)^n (f \otimes \overline{f}) \right\|_{L^2(\mu\times\mu)} \leq \|f\|^2_{U^2(X,\mu,T)}.$$

Now to apply Lemma 5.4, we can write $(\mathbf{1} \otimes f \otimes \mathbf{1}) \otimes \overline{(\mathbf{1} \otimes f \otimes \mathbf{1})} = (f \otimes \overline{f}) \circ (\pi_1 \times \pi_1)$ and $\nu = (\pi_1 \times \pi_1)_*(\tau \times \tau)$ to obtain

$$\int_{X\times X} f \otimes \overline{f} d(\mu \times \mu) = \int_{X^3\times X^3} (f \otimes \overline{f}) \circ (\pi_1 \times \pi_1) d(\tau \times \tau) = \int_{X^3\times X^3} (\mathbf{1} \otimes f \otimes \mathbf{1}) \otimes \overline{(\mathbf{1} \otimes f \otimes \mathbf{1})} d(\tau \times \tau),$$

and so for each N, we have

$$\begin{aligned} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} T^n f \otimes \overline{T^n f} \right\|_{L^2(\mu\times\mu)} &= \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (\mathbf{1} \otimes T^n f \otimes \mathbf{1}) \otimes \overline{(\mathbf{1} \otimes T^n f \otimes \mathbf{1})} \right\|_{L^2(\tau\times\tau)} \\ &\overset{(5.4)}{\geq} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (\mathbf{1} \otimes T^n f \otimes \mathbf{1}) \right\|^2_{L^2(\tau)}. \end{aligned}$$

Taking the limit superior of both sides and using Lemma 5.6 part (a), we finally arrive at

$$\|f\|^2_{U^2(X,\mu,T)} \geq \limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} T^n f \otimes \overline{T^n f} \right\|_{L^2(\mu\times\mu)} \geq \limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (\mathbf{1} \otimes T^n f \otimes \mathbf{1}) \right\|^2_{L^2(\tau)}.$$

We are essentially done now, as if $b : \mathbb{N} \to \mathbb{C}$ is bounded, then

$$\begin{aligned} \limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} b(n)(\mathbf{1} \otimes T^n f \otimes \mathbf{1}) \right\|_{L^2(\tau)} &\leq \limsup_{N\to\infty} \|b\|_\infty \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (\mathbf{1} \otimes T^n f \otimes \mathbf{1}) \right\|_{L^2(\tau)} \\ &\leq \|b\|_\infty \|f\|_{U^2(X,\mu,T)}. \end{aligned}$$

■

The following lemma is the final step in the proof of Theorem 5.2.

Lemma 5.9. *Let (X, μ, T) be an ergodic system, $a \in \mathbf{gen}(\mu, \Phi)$, (Z_1, m, R) the Kronecker factor of (X, μ, T). Let (Y, S) be another Kronecker system, then for any $g \in C(X), y \in Y, F \in C(Y)$,*

$$\limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} g(T^n a) F(S^n y) \right| \leq \|E(g|\mathscr{Z}_1)\|_{L^1(m_{Z_1})} \cdot \|F\|_\infty. \tag{5.9}$$

Proof. Pick a subequence (N_j) such that

(a)

$$\lim_{j\to\infty} \left| \frac{1}{|\Phi_{N_j}|} \sum_{n\in\Phi_{N_j}} g(T^n a) F(S^n y) \right| = \limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} g(T^n a) F(S^n y) \right|.$$

(b) $(a, y) \in \mathbf{gen}(\rho, (\Phi_{N_j})_j)$ for some $T \times S$-invariant measure ρ on $X \times Y$.

Note that the second marginal of ρ must be m_Y, as ρ_2 is S-invariant and Y is a Kronecker system, so it is uniquely ergodic. It also follows that the first marginal of ρ is equal to μ since a is generic for μ along Φ and (a, y) is generic for ρ also along a subsequence of Φ.

Now write $\tilde{g} := E_\mu(g|\mathscr{Z}_1) \circ \pi$, where $\pi : X \to Z_1$ is the factor map from X to its Kronecker factor,

$$\lim_{j\to\infty} \left| \frac{1}{|\Phi_{N_j}|} \sum_{n\in\Phi_{N_j}} g(T^n a)F(S^n y) \right| = \left| \int_{X\times Y} g \otimes F d\rho \right| \leq \left| \int_{X\times Y} (g - \tilde{g}) \otimes F d\rho \right| + \left| \int_{X\times Y} \tilde{g} \otimes F d\rho \right|. \tag{5.9.1}$$

Since $E_\mu(g-\tilde{g}|\mathscr{Z}_1)\circ\pi = 0$ μ-a.e., we claim then $E_\rho((g-\tilde{g})\otimes\mathbf{1}|\mathscr{U}_1)\circ\varphi = 0$ ρ-a.e., where $\varphi : X\times Y \to U_1$ is the factor map to the Kronecker factor and $\mathscr{U}_1$ is the σ-algebra corresponding to the Kronecker factor of $(X \times Y, \rho, T \times S)$. Let $\mathscr{N} = \{\varnothing, Y\}$ be the trivial σ-algebra on Y, and observe that $\mathscr{Z}_1 \otimes \mathscr{N} \subset \mathscr{U}$, as if f is a μ-eigenfunction for T, then $f \otimes \mathbf{1}$ is a $T \times S$ ρ-eigenfunction, as $(\pi_1)_*\rho = \mu$, so we get $f \otimes \mathbf{1} \circ T \times S = \lambda f \otimes \mathbf{1}$ ρ-a.e.

Since $\mathscr{U}_1$ is the σ-algebra generated by $T \times S$ ρ-eigenfunctions, and $\mathscr{Z}_1$ is generated by T μ-eigenfunctions, then we have that every generating set of $\mathscr{Z}_1 \otimes \mathscr{N}$ is $\mathscr{U}_1$-measurable. Also observe that since ρ is a joining,

$$E_\rho((g - \tilde{g}) \otimes \mathbf{1}|\mathscr{Z}_1 \otimes \mathscr{N}) = E_\mu(g - \tilde{g}|\mathscr{Z}_1) \otimes E_{m_Y}(\mathbf{1}|\mathscr{N}) = E_\mu(g - \tilde{g}|\mathscr{Z}_1) \otimes \mathbf{1}\ \rho - \text{a.e.}$$

All of this gives the following, where equalities are ρ-a.e.

$$0 = E_\rho(E_\mu(g - \tilde{g}|\mathscr{Z}_1) \otimes \mathbf{1}|\mathscr{U}_1) = E_\rho(E_\rho((g - \tilde{g}) \otimes \mathbf{1}|\mathscr{Z}_1 \otimes \mathscr{N})|\mathscr{U}) \overset{\mathscr{Z}_1\otimes\mathscr{N}\subset\mathscr{U}}{=} E_\rho((g - \tilde{g}) \otimes \mathbf{1}|\mathscr{U}_1).$$

Moreover, $\mathbf{1} \otimes F$ is $\mathscr{U}_1$-measurable, as Y is a Kronecker system and $F \in C(Y)$, so $E_\rho(\mathbf{1} \otimes F|\mathscr{U}_1) = \mathbf{1} \otimes F$. Viewing $E_\rho(\cdot|\mathscr{U}_1)$ as projection onto the subspace in $L^2(X \times Y, \rho)$ of $\mathscr{U}_1$-measurable functions, we see that $(g - \tilde{g}) \otimes \mathbf{1}$ and $\mathbf{1} \otimes F$ are orthogonal, as $E_\rho((g - \tilde{g}) \otimes \mathbf{1}|\mathscr{U}_1) = 0$ while $E_\rho(\mathbf{1} \otimes F|\mathscr{U}_1) = \mathbf{1} \otimes F$. So $\int_{X\times Y}(g - \tilde{g}) \otimes F d\rho = 0$.

We are left to understand the final term, $\int_{X\times Y} \tilde{g} \otimes F d\rho$:

$$\left| \int_{X\times Y} \tilde{g} \otimes F d\rho \right| \leq \|F\|_\infty \cdot \left| \int_{X\times Y} \tilde{g} \otimes \mathbf{1} d\rho \right| = \|F\|_\infty \cdot \left| \int_X \tilde{g} d\mu \right| = \|F\|_\infty \|E(g|\mathscr{Z}_1)\|_{L^1(m_{Z_1})}. \tag{5.9.2}$$

Combining (5.9.1) and (5.9.2), we have what we wanted. ■

Finally we move to the proof of Theorem 5.2.

Proof of Theorem 5.2. By Stone-Weierstrass, we can show the statement for all functions of the form $f_1 \otimes f_2$, where $f_i \in C(X)$ and $0 \leq f_i \leq 1$, in place of $G \in C(X^2)$. We wish to show the following

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T^n_\Delta(f_1 \otimes f_2 \otimes \mathbf{1}) - T^n_\Delta(\mathbf{1} \otimes f_1 \otimes f_2)) \right\|_{L^2(\sigma)} = 0.$$

Write $g_i = E(f_i|\mathscr{Z}_1) \circ \pi$, and observe that $T^n_\Delta(f_1 \otimes f_2 \otimes \mathbf{1}) = f_1(T^n a)(\mathbf{1} \otimes f_2 \otimes \mathbf{1})$ σ-a.e. By applying the triangle inequality a few times, it suffices to show each of the following quantities are zero.

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a)T^n_\Delta(\mathbf{1} \otimes f_2 \otimes \mathbf{1}) - f_1(T^n a)T^n_\Delta(\mathbf{1} \otimes g_2 \otimes \mathbf{1}) \right\|_{L^2(\sigma)}, \tag{5.2.1}$$

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a)T^n_\Delta(\mathbf{1} \otimes g_2 \otimes \mathbf{1}) - T^n_\Delta(\mathbf{1} \otimes g_1 \otimes g_2) \right\|_{L^2(\sigma)}, \tag{5.2.2}$$

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (T^n_\Delta(\mathbf{1} \otimes f_1 \otimes f_2) - T^n_\Delta(\mathbf{1} \otimes g_1 \otimes g_2)) \right\|_{L^2(\sigma)}. \tag{5.2.3}$$

(5.2.1) is zero by Lemma 5.8 and (5.2.3) is also zero by applying Lemma 5.7, as $E(f_i - g_i|\mathscr{Z}_1) = 0$ almost everywhere, which is equivalent to $\|f_i - g_i\|_{U^2(X,\mu,T)} = 0$. For (5.2.2), we must do some more work, specifically we will approximate $E(f|\mathscr{Z}_1)$ with continuous functions. Note that showing (5.2.2) is zero is equivalent to proving the analgous statement for the measure $\xi \in \mathcal{M}(Z_1^2)$, specifically

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a) T_\Delta^n(E(f_2|\mathscr{Z}_1)\otimes \mathbf{1}) - T_\Delta^n(E(f_1|\mathscr{Z}_1)\otimes E(f_2|\mathscr{Z}_1)) \right\|_{L^2(\xi)} = 0. \tag{5.2.4}$$

Let $\varepsilon \in (0,1)$, and for each $i = 1,2$, let $h_i \in C(Z_1)$ so that $\|E(f_i|\mathscr{Z}_1) - h_i\|_{L^2(m_{Z_1})} < \varepsilon$. We can assume further that $0 \le h_i \le 1$, since $0 \le f_i \le 1$, then $0 \le E(f_i|\mathscr{Z}_1) \le 1$ μ-a.e., as if $E_1 = \{x : E(f_i|\mathscr{Z}_1)(x) > 1\} \in \mathscr{Z}_1$ has $\mu(E_1) > 0$, then $m_{Z_1}(E_1) < \int_{E_1} E(f_i|\mathscr{Z}_1) dm_{Z_1} = \int_{\pi_1^{-1}E_1} f_i d\mu \le \mu(\pi_1^{-1}E_1) = m_{Z_1}(E_1)$, a contradiction. Similarly $E(f_i|\mathscr{Z}_1) \ge 0$ a.e. With this in mind, we can let $h_i' \in C(Z_1)$ so that $\|E(f|\mathscr{Z}_1) - h_i'\|_{L^2(m_{Z_1})} < \varepsilon$, and we can define $h_i(x) = h_i'(x)$ if $0 \le h_i'(x) \le 1$, otherwise $h_i(x) = 0$ if $h_i'(x) < 0$ and $h_i(x) = 1$ when $h_i'(x) > 1$. h_i will be a better approximation of $E(f_i|\mathscr{Z}_1)$ in L^2, will still be continuous, and takes values in $[0,1]$.

With this bounded, continuous approximation, we will now show three statements to prove (5.2.4).

$$\limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a) T_\Delta^n(E(f_2|\mathscr{Z}_1)\otimes \mathbf{1}) - h_1(\pi_1(T^n a)) T_\Delta^n(h_2\otimes \mathbf{1}) \right\|_{L^2(\xi)} < \varepsilon + \sqrt{\varepsilon}, \tag{5.2.5}$$

$$\lim_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} h_1(\pi_1(T^n a)) T_\Delta^n(h_2\otimes \mathbf{1}) - T_\Delta^n(h_1\otimes h_2) \right\|_{L^2(\xi)} = 0, \tag{5.2.6}$$

$$\limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} T_\Delta^n(h_1\otimes h_2) - T_\Delta^n(E(f_1|\mathscr{Z}_1)\otimes E(f_2|\mathscr{Z}_1)) \right\|_{L^2(\xi)} < \sqrt[4]{2}\varepsilon. \tag{5.2.7}$$

(5.2.5) will be proved using both L^2 approximation and Lemma 5.9, (5.2.6) will be shown using the definition of ξ, and (5.2.7) will follow by L^2-approximation.

(5.2.7): Recall that both $h_i, E(f_i|\mathscr{Z}_1)$ take values in $[0,1]$ for m_{Z_1}-almost every $z \in Z_1$. Let $A \subset Z_1$ be the set of all z such that $(h_1 - E(f_1|\mathscr{Z}_1))(z) \in [-1,1]$, then $m_{Z_1}(A) = 1$, and letting B be the same set for $h_2 - E(f_2|\mathscr{Z}_1)$, then B is also full measure. Since $\xi_1 = m_{Z_1}$, and $\xi_2 \le 2m_{Z_1}$. Now $\xi(A\times Z_1) = \xi_1(A) = m_{Z_1}(A) = 1$, and $\xi((Z_1\times B)^c) = \xi(Z_1\times B^c) = \xi_2(B^c) \le 2m_{Z_1}(B^c) = 0$, so $\xi(Z_1\times B) = 1$, thus $\xi((A\times Z_1)\cap(Z_1\times B)) = \xi(A\times Z_1)\xi(Z_1\times B) = \xi(A\times B) = 1$. All of this to say that $(h_1 - E(f_1|\mathscr{Z}_1))\otimes(h_2 - E(f_2|\mathscr{Z}_1))$ takes values in $[-1,1]$ ξ-a.e. This allows us to make the following estimate for each $k \in \mathbb{N}$.

$$\int_{Z^2} |T_\Delta^k(h_1\otimes h_2) - T_\Delta^k(E(f_1|\mathscr{Z}_1)\otimes E(f_2|\mathscr{Z}_1))|^2 d\xi \le \int_{Z^2} |T_\Delta^k(h_1\otimes h_2) - T_\Delta^k(E(f_1|\mathscr{Z}_1)\otimes E(f_2|\mathscr{Z}_1))| d\xi$$

Now by Cauchy-Schwarz and $\xi_i \le i m_{Z_1}$, we have by T-invariance of m_{Z_1}

$$\begin{aligned}
\|T_\Delta^k(h_1\otimes h_2) - T_\Delta^k(E(f_1|\mathscr{Z}_1)\otimes E(f_2|\mathscr{Z}_1))\|_{L^1(\xi)} &\le \|T^k(h_1 - E(f_1|\mathscr{Z}_1))\|_{L^2(\xi_1)} \cdot \|T^k(h_2 - E(f_2|\mathscr{Z}_1))\|_{L^2(\xi_2)} \\
&\le \|T^k(h_1 - E(f_1|\mathscr{Z}_1))\|_{L^2(m_{Z_1})} \cdot \sqrt{2}\|T^k(h_2 - E(f_2|\mathscr{Z}_1))\|_{L^2(m_{Z_1})} \\
&= \|h_1 - E(f_1|\mathscr{Z}_1)\|_{L^2(m_{Z_1})} \cdot \sqrt{2}\|h_2 - E(f_2|\mathscr{Z}_1)\|_{L^2(m_{Z_1})} < \sqrt{2}\varepsilon^2.
\end{aligned}$$

Thus we have for each $k \in \mathbb{N}$,

$$\|T_\Delta^k(h_1\otimes h_2) - T_\Delta^k(E(f_1|\mathscr{Z}_1)\otimes E(f_2|\mathscr{Z}_1))\|_{L^2(\xi)}^2 < \sqrt{2}\varepsilon^2.$$

Then by applying the triangle inequality to each term in (5.2.7) and taking the limit superior, we get what was desired (since each term in the average has L^2-norm less than $\sqrt[4]{2}\varepsilon$).

(5.2.5): Apply the triangle inequality once again to obtain

$$\limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a) T_\Delta^n(E(f_2|\mathscr{Z}_1)\otimes \mathbf{1}) - h_1(\pi_1(T^n a)) T_\Delta^n(h_2\otimes \mathbf{1}) \right\|_{L^2(\xi)} \le$$

$$\limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a) T_\Delta^n(E(f_2|\mathscr{Z}_1)\otimes \mathbf{1}) - f_1(T^n a) T_\Delta^n(h_2\otimes \mathbf{1}) \right\|_{L^2(\xi)} + \tag{5.2.8}$$

$$\text{(5.2.9)} \qquad \limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} f_1(T^n a) T^n_\Delta(h_2 \otimes \mathbf{1}) - h_1(\pi_1(T^n a)) T^n_\Delta(h_2 \otimes \mathbf{1}) \right\|_{L^2(\xi)} .$$

By the definition of h_2, ξ_1, and T-invariance of m_{Z_1}, we have that for each k, $\|T^k_\Delta(E(f_2|\mathscr{Z}_1)\otimes\mathbf{1}) - T^k_\Delta(h_2\otimes\mathbf{1})\|_{L^2(\xi)} = \|T^k(E(f_2|\mathscr{Z}_1) - h_2)\|_{L^2(\xi_1)} < \varepsilon$. Since $0 \le f_1 \le 1$, then

$$\|f_1(T^k a)T^k_\Delta(E(f_2|\mathscr{Z}_1)\otimes\mathbf{1}) - f_1(T^k a)T^k_\Delta(h_2\otimes\mathbf{1})\|_{L^2(\xi)} \le \|T^k_\Delta(E(f_2|\mathscr{Z}_1)\otimes\mathbf{1}) - T^k_\Delta(h_2\otimes\mathbf{1})\|_{L^2(\xi)} < \varepsilon.$$

So (5.2.8) is bounded by ε by averaging.

We will apply Lemma 5.9 to (5.2.9). Since f_1, h_1, π_1 are all continuous, then $f_1 - h_1 \circ \pi_1$ is continuous on X, and $a \in \mathbf{gen}(\mu, \Phi)$, $h_1 \in C(Z_1)$, then an application of Lemma 5.9 seems close. Now $\|f_1 - h_1 \circ \pi_1\|_\infty \le 1$ and $0 \le h_2 \le 1$ on all of X, Z_1 respectively, so for each $N \in \mathbb{N}$,

$$\left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (f_1(T^k a) - h_1(\pi_1(T^k a))) T^k_\Delta(h_2\otimes\mathbf{1}) \right\|^2_{L^2(\xi)} \le \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (f_1(T^k a) - h_1(\pi_1(T^k a))) T^k_\Delta(h_2\otimes\mathbf{1}) \right\|_{L^1(\xi)}$$

since the integrand on the LHS is bounded above by 1 everywhere. With this, we extract the following

$$\begin{aligned}
&\limsup_{N\to\infty} \left\| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} (f_1(T^k a) - h_1(\pi_1(T^k a))) T^k_\Delta(h_2\otimes\mathbf{1}) \right\|_{L^1(\xi)} = \\
&\limsup_{N\to\infty} \int_{Z_1^2} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} ((f_1 - h_1\circ\pi_1)(T^k a)) T^k_\Delta(h_2\otimes\mathbf{1}) \right| d\xi \le \\
&\int_{Z_1^2} \limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} ((f_1 - h_1\circ\pi_1)(T^k a)) T^k_\Delta(h_2\otimes\mathbf{1}) \right| d\xi \le \\
&\sup_{(x,y)\in Z_1^2} \limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} ((f_1 - h_1\circ\pi_1)(T^k a))(h_2\otimes\mathbf{1})(T^k x, T^k y) \right|.
\end{aligned}$$

Now $(Z_1 \times Z_1, T \times T)$ is a Kronecker system and $h_2 \otimes \mathbf{1} \in C(Z_1^2)$, so by Lemma 5.9, for any $(x, y) \in Z_1^2$,

$$\begin{aligned}
&\limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} ((f_1 - h_1\circ\pi_1)(T^k a))(h_2\otimes\mathbf{1})(T^k x, T^k y) \right| \le \\
&\|E(f_1 - h_1\circ\pi_1|\mathscr{Z}_1)\|_{L^1(m_{Z_1})} \cdot \|h\otimes\mathbf{1}\|_\infty \le \|E(f_1|\mathscr{Z}_1) - h_1\|_{L^2(m_{Z_1})} \cdot 1 < \varepsilon.
\end{aligned}$$

Thus

$$\sup_{(x,y)\in Z_1^2} \limsup_{N\to\infty} \left| \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} ((f_1 - h_1\circ\pi_1)(T^k a))(h_2\otimes\mathbf{1})(T^k x, T^k y) \right| \le \varepsilon,$$

and with that we have the desired bound for (5.2.5).

(5.2.6): We will use the definition of ξ, recall that we defined

$$\xi := \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} \delta_{T^n\pi_1(a)} \times \delta_{T^{2n}\pi_1(a)}$$

for a point $a \in \mathbf{gen}(\mu, \Phi)$, with this limit being taken in the sense of weak-$\star$ convergence. This limit is the same regardless of the Følner sequence it is taken over. In fact, ξ is the unique $T \times T^2$-invariant measure on the orbit closure of $(\pi_1(a), \pi_1(a)) \in Z_1^2$ under $T \times T^2$. All of this will be crucial in proving (5.2.6).

Since $\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(\pi_1(T^n a))T^n_\Delta(h_2\otimes\mathbf{1}) - T^n_\Delta(h_1\otimes h_2)$ is continuous for each $N\in\mathbb{N}$, we can apply the definition of ξ using the Følner sequence Φ to get

(5.2.10)

$$\lim_{N\to\infty}\left\|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(\pi_1(T^n a))T^n_\Delta(h_2\otimes\mathbf{1}) - T^n_\Delta(h_1\otimes h_2)\right\|^2_{L^2(\xi)} =$$

$$\lim_{N\to\infty}\int_{Z_1^2}\left|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(\pi_1(T^n a))(T^n h_2\otimes\mathbf{1}) - \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}(T^n h_1\otimes T^n h_2)\right|^2 d\xi =$$

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{|\Phi_M|}\sum_{m\in\Phi_M}\left|\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(\pi_1(T^n a))h_2(\pi_1(T^{n+m}a)) - \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(\pi_1(T^{n+m}a)\cdot h_2(\pi_1(T^{n+2m}a))\right|^2.$$

Let $\tilde a = \pi_1(a)$, and let

$$\nu_N = \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\delta_{T^n_\Delta(\tilde a,\tilde a)}, \nu = \lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}\delta_{T^n_\Delta(\tilde a,\tilde a)} = (R_{(\tilde a,\tilde a)})_* m_\Delta,$$

where m_Δ is Haar measure on $\{(z,z) : z\in Z_1\}\leqslant Z_1\times Z_1$. This limit exists and is independent of Følner sequence as argued in Proposition 4.5. Let $S = T\times T^2$. Note that for any $m\in\mathbb{N}$,

$$\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(T^n\tilde a)h_2(T^{n+m}\tilde a) = \lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1\otimes h_2((T^{n-m}_\Delta\circ S^m)(\tilde a,\tilde a)) \to \int_{Z_1^2} h_1\otimes h_2\circ(T^{-m}_\Delta\circ S^m)d\nu$$

$$(\nu \text{ is } T_\Delta\text{-invariant, } T_\Delta \text{ and } S \text{ commute}) = \int_{Z_1^2} h_1\otimes h_2\circ S^m d\nu.$$

$$\lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1(T^{n+m}\tilde a)h_2(T^{n+2m}\tilde a) = \lim_{N\to\infty}\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} h_1\otimes h_2((T^n_\Delta\circ S^m)(\tilde a,\tilde a)) \to \int_{Z_1^2} h_1\otimes h_2\circ S^m d\nu.$$

Let $f_m = h_1\otimes h_2\circ S^m$, we claim $\{f_m\}_{m\in\mathbb{N}}$ is an equicontinuous family of functions on Z_1^2. Let d be a Z_1^2-invariant metric on Z_1^2. Now if α is the element defining the rotation T, then $S(z,w) = (z+\alpha, w+2\alpha)$, so S is an isometry of Z_1^2. Thus $\{f_m\}_{m\in\mathbb{N}}$ is an equicontinuous family of functions.

To see this, let $\varepsilon>0$, then there is a $\delta>0$ so that $d(x,y)<\delta\Rightarrow d(f_0(x),f_0(y))<\varepsilon$, as Z_1^2 is compact, so f_0 is uniformly continuous. We claim this δ works for all f_m, as $d(f_m(x),f_m(y)) = d(f_0\circ S^m(x), f_0\circ S^m(y)) = d(f_0(x),f_0(y))<\varepsilon$, thus we have an equicontinuous family. Then by the Arzelà-Ascoli theorem, since $\{f_m\}_{m\in\mathbb{N}}$ is uniformly bounded by 1 and equicontinuous in $C(Z_1^2)$, then this family is precompact in the uniform norm topology on $C(Z_1^2)$.

Note $\nu_N \xrightarrow{*} \nu$ in the weak-* topology. Now for any $\varepsilon>0$, there is an ε-dense set $g_1,\dots,g_k\in\overline{\{f_m\}_{m\in\mathbb{N}}}$. With this, we obtain uniform convergence in m for $\int f_m d\nu_N\to\int f_m d\nu$. To see this, let N be such that $n\geq N\Rightarrow |\int g_i d\nu_n - \int g_i d\nu|<\varepsilon$ for each $i=1,\dots,k$, then for any m, there is g_i so that $\|f_m-g_i\|_\infty<\varepsilon$, and thus

$$\left|\int f_m d\nu_n - \int f_m d\nu\right| \leq \left|\int f_m d\nu_n - \int g_i d\nu_n\right| + \left|\int g_i d\nu_n - \int g_i d\nu\right| + \left|\int g_i d\nu - \int f_m d\nu\right| < 3\varepsilon.$$

In other words,

$$\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}(h_1\otimes h_2\circ S^m)((T^n_\Delta(\tilde a,\tilde a)) \to \int h_1\otimes h_2\circ S^m d\nu,$$

$$\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}(h_1\otimes h_2\circ S^m)((T^{n-m}_\Delta(\tilde a,\tilde a)) \to \int h_1\otimes h_2\circ S^m d\nu$$

uniformly in m. With this, we can show (5.2.10) is zero. Let $\varepsilon > 0$, there is K so that $N \geq K$ gives for each m

$$\left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} (h_1 \otimes h_2 \circ S^m)((T_\Delta^n(\tilde{a}, \tilde{a})) - \int h_1 \otimes h_2 \circ S^m d\nu \right| < \varepsilon,$$

$$\left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} (h_1 \otimes h_2 \circ S^m)((T_\Delta^{n-m}(\tilde{a}, \tilde{a})) - \int h_1 \otimes h_2 \circ S^m d\nu \right| < \varepsilon.$$

Then, investigating the quantity in (5.2.10), for $N \geq K$ and any M,

$$\frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} h_1(\pi_1(T^n a)) h_2(\pi_1(T^{n+m} a)) - \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} h_1(\pi_1(T^{n+m} a) \cdot h_2(\pi_1(T^{n+2m} a)) \right|^2 \leq$$

$$\frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \left(\begin{array}{l} \left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} h_1(\pi_1(T^n a)) h_2(\pi_1(T^{n+m} a)) - \int h_1 \otimes h_2 \circ S^m \, d\nu \right| \\ \quad + \left| \int h_1 \otimes h_2 \circ S^m \, d\nu - \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} h_1(\pi_1(T^{n+m} a)) \cdot h_2(\pi_1(T^{n+2m} a)) \right| \end{array} \right)^2 <$$

$$\frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} 4\varepsilon^2 = 4\varepsilon^2.$$

Thus for $N \geq K$,

$$\lim_{M \to \infty} \frac{1}{|\Phi_M|} \sum_{m \in \Phi_M} \left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} h_1(\pi_1(T^n a)) h_2(\pi_1(T^{n+m} a)) - \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} h_1(\pi_1(T^{n+m} a) \cdot h_2(\pi_1(T^{n+2m} a)) \right|^2 \leq 2\varepsilon^2.$$

It follows that the double limit in (5.2.10) is zero. ■

Appendix A. Følner Results

We collect some technical results concerning Følner sets used earlier in the paper.

Lemma A.1. *Let Φ_N be a Følner sequence, and define $\Psi_N := \{n \in \mathbb{N} : 2n \in \Phi_N\}$, $E_N := \Phi_N \cap 2\mathbb{N}, O_N := \Phi_N \cap (2\mathbb{N} + 1)$. Then*

$$\lim_{N \to \infty} \frac{|\Psi_N|}{|\Phi_N|} = \lim_{N \to \infty} \frac{|E_N|}{|\Phi_N|} = \frac{1}{2}.$$

Proof. Observe that $|\Psi_N| = |E_N|$, so we will just prove the statement for the second limit. Define $F_N := \{n \in E_N : n - 1 \in O_N\}, G_N := \{n \in O_N : n - 1 \in E_N\}$, let $\varepsilon > 0$ and let N be such that $n \geq N$ implies

$$\frac{|(\Phi_n + 1) \cap \Phi_n|}{|\Phi_n|} > 1 - \varepsilon.$$

Now observe that $E_n \setminus F_n = \{n \in E_n : n - 1 \notin \Phi_n\} \subset \Phi_n \setminus (\Phi_n + 1)$, so $|E_n \setminus F_n| \leq \varepsilon |\Phi_n|$, giving that

$$\frac{|F_n|}{|E_n|} = \frac{|E_n| - |E_n \setminus F_n|}{|E_n|} \geq 1 - \varepsilon \frac{|\Phi_n|}{|E_n|},$$
$$|F_n| \geq |E_n| - \varepsilon |\Phi_n|.$$

A symmetric argument yields that $|G_n| \geq |O_n| - \varepsilon |\Phi_n|$. Since $F_n = E_n \cap (O_n + 1)$, $G_n = O_n \cap (E_n + 1)$, then both

$$\frac{|F_n|}{|\Phi_n|}, \frac{|G_n|}{|\Phi_n|} \leq \frac{1}{2},$$

as if not, since the even and odd elements partition Φ_n, we would get

$$\frac{|F_n|}{|\Phi_n|} > \frac{1}{2}, \frac{|E_n|}{|\Phi_n|} \geq \frac{|F_n|}{|\Phi_n|} \Rightarrow \frac{|E_n|}{|\Phi_n|} > \frac{1}{2}.$$
$$\frac{|E_n| + |O_n|}{|\Phi_n|} = 1 \Rightarrow \frac{|O_n|}{|\Phi_n|} < \frac{1}{2}. \text{ But then } \frac{|F_n|}{|\Phi_n|} \leq \frac{|O_n|}{|\Phi_n|} < \frac{1}{2}.$$

This yields a contradiction, as does the case where $\frac{|G_n|}{|\Phi_n|} > \frac{1}{2}$. Putting all of this together, we have

$$\frac{1}{2} - \varepsilon \leq \frac{|E_n|}{|\Phi_n|} - \varepsilon \leq \frac{|F_n|}{|\Phi_n|} \leq \frac{1}{2},$$
$$\frac{1}{2} - \varepsilon \leq \frac{|O_n|}{|\Phi_n|} - \varepsilon \leq \frac{|G_n|}{|\Phi_n|} \leq \frac{1}{2}.$$

So each of $\frac{|F_n|}{|\Phi_n|}, \frac{|G_n|}{|\Phi_n|}$ goes to $\frac{1}{2}$, but more importantly we extract that

$$\frac{1}{2} - 2\varepsilon \leq \frac{|E_n|}{|\Phi_n|} \leq \frac{1}{2} + \varepsilon, \frac{1}{2} - 2\varepsilon \leq \frac{|O_n|}{|\Phi_n|} \leq \frac{1}{2} + \varepsilon,$$

so each of these ratios also converges to $\frac{1}{2}$. ■

Lemma A.2. *Let Φ_N be a Følner sequence, Ψ_N, Θ_N as defined in Lemma 5.7, then Ψ_N, Θ_N each form Følner sequences.*

Proof. E_N, O_N are defined as in Lemma A.1.

We show Ψ_N form a Følner sequence. Notice that $n \in \Psi_N \cap (\Psi_N + 1) \iff n, n-1 \in \Psi_N \iff 2n, 2n-2 \in \Phi_N \iff 2n \in E_N \cap (E_N + 2)$, so $|\Psi_N \cap (\Psi_N + 1)| = |E_N \cap (E_N + 2)|$ for each N. Let $N \in \mathbb{N}$, then

$$\frac{|E_N \cap (E_N + 2)|}{|\Phi_N|} \leq \frac{|E_N|}{|\Phi_N|} \Rightarrow \limsup_{N\to\infty} \frac{|E_N \cap (E_N + 2)|}{|\Phi_N|} \leq \frac{1}{2}. \tag{A.2.1}$$

We will prove the opposite direction of this inequality for the limit inferior. Observe that $E_N \triangle (E_N + 2) \subset \Phi_N \triangle (\Phi_N + 2)$. We take care to explicitly note this, as the inclusion fails if one considers $E_N \triangle (E_N + 1) = E_N \cup (E_N + 1)$. Let $\varepsilon > 0$, and let N be such that for $n \geq N$, we have

$$\begin{aligned}
&\frac{|(\Phi_n + 2)\triangle\Phi_n|}{|\Phi_n|} < \varepsilon, \ \left|\frac{|E_n|}{|\Phi_n|} - \frac{1}{2}\right| < \varepsilon \\
&\Rightarrow \frac{|E_n|}{|\Phi_n|} = \frac{|E_n \cap (E_n + 2)|}{|\Phi_n|} + \frac{|E_n \triangle (E_n + 2)|}{|\Phi_n|} \leq \frac{|E_n \cap (E_n + 2)|}{|\Phi_n|} + \frac{|(\Phi_n + 2)\triangle\Phi_n|}{|\Phi_n|} < \frac{|E_n \cap (E_n + 2)|}{|\Phi_n|} + \varepsilon \\
&\Rightarrow n \geq N \Rightarrow \frac{1}{2} - \varepsilon < \frac{|E_n|}{|\Phi_n|} < \frac{|E_n \cap (E_n + 2)|}{|\Phi_n|} + \varepsilon \Rightarrow \frac{1}{2} - 2\varepsilon < \frac{|E_n \cap (E_n + 2)|}{|\Phi_n|}.
\end{aligned}$$

So for each $\varepsilon > 0$, there is an N after which $\frac{|E_n \cap (E_n+2)|}{|\Phi_n|} > \frac{1}{2} - 2\varepsilon$, thus

$$\liminf_{N\to\infty} \frac{|E_n \cap (E_n + 2)|}{|\Phi_n|} \geq \frac{1}{2}. \tag{A.2.2}$$

Putting (A.2.1) and (A.2.2) together, we see that

$$\lim_{N\to\infty} \frac{|\Psi_N \cap (\Psi_N + 1)|}{|\Phi_N|} = \lim_{N\to\infty} \frac{|E_N \cap (E_N + 2)|}{|\Phi_N|} = \frac{1}{2}. \tag{A.2.3}$$

We know from Lemma A.1 that

$$\lim_{N\to\infty} \frac{|\Psi_N|}{|\Phi_N|} = \lim_{N\to\infty} \frac{|E_N|}{|\Phi_N|} = \frac{1}{2},$$

so combining this with (A.2.3),

$$\underbrace{\left(\lim_{N\to\infty} \frac{|\Psi_N \cap (\Psi_N + 1)|}{|\Phi_N|}\right)}_{=\frac{1}{2}} \underbrace{\left(\lim_{N\to\infty} \frac{|\Phi_N|}{|\Psi_N|}\right)}_{=2} = \lim_{N\to\infty} \frac{|\Psi_N \cap (\Psi_N + 1)|}{|\Psi_N|} = 1.$$

So Ψ_N forms a Følner sequence. Since $|\Theta_N \cap (\Theta_N + 1)| = |O_N \cap (O_N + 1)|$, an identical argument will show that Θ_N also forms a Følner sequence. ■

References

[EW10] Manfred Einsiedler and Thomas Ward. *Ergodic Theory with a view towards Number Theory*. Springer London, 2010.

[FH16] Nikos Frantzikinakis and Bernard Host. Weighted multiple ergodic averages and correlation sequences. *Ergodic Theory and Dynamical Systems*, 38(1):81–142, July 2016.

[FK24] Nikos Frantzikinakis and Borys Kuca. Joint ergodicity for commuting transformations and applications to polynomial sequences. *Inventiones Mathematicae*, 2024.

[Fur77] Harry Furstenberg. Ergodic behavior of diagonal measures and a theorem of szemerédi on arithmetic progressions. *Journal d'Analyse Mathématique*, 31:204–256, 1977.

[Fur81] Harry Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory*. Princeton University Press, 1981.

[Hin74] Neil Hindman. Finite sums from sequences within cells of a partition of n. *Journal of Combinatorial Theory, Series A*, 17(1):1–11, 1974.

[Hin79] Neil Hindman. *Ultrafilters and combinatorial number theory*, pages 119–184. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979.

[HK18] Bernard Host and Bryna Kra. *Nilpotent Structures in Ergodic Theory*. American Mathematical Society, 2018.

[Hos08] Bernard Host. Ergodic seminorms for commuting transformations and applications. *Studia Mathematica*, 2008.

[KMRR23] Bryna Kra, Joel Moreira, Florian Richter, and Donald Robertson. Infinite sumsets in sets with positive density. *Journal of the American Mathematical Society*, 37(3):637–682, August 2023.

[KMRR26] Bryna Kra, Joel Moreira, Florian Richter, and Donald Robertson. The density finite sums theorem. *Inventiones Mathematicae*, 2026.

[Kra06] Bryna Kra. Ergodic methods in additive combinatorics, 2006.

[MRR19] Joel Moreira, Florian Richter, and Donald Robertson. A proof of a sumset conjecture of erdős. *Annals of Mathematics*, 189(2), March 2019.

[Rot53] K. F. Roth. On certain sets of integers. *Journal of the London Mathematical Society*, s1-28(1):104–109, 1953.

[Sze75] Endre Szemerédi. On sets of integers containing k elements in arithmetic progression. *Acta Arithmetica*, 27(1):199–245, 1975.

[vdWBL27] van der WAERDEN B. L. Beweis einer baudetschen vermutung. *Nieuw Arch. Wiskunde*, 15:212–216, 1927.